**F6.** If $C$ and $C'$ are closed cones, and if $C^* + C'^*$ is closed, we have $(C \cap C')^* = C^* + C'^*$.

**Proof.** If $A$ and $B$ are arbitrary sets we have $A^* + B^* \subseteq (A \cap B)^*$, for if $x \in A^* + B^*$ and $y \in A \cap B$ then $x \cdot y = a \cdot y + b \cdot y \geq 0$. If $A$ and $B$ are arbitrary sets including 0 then $(A + B)^* \subseteq A^* \cap B^*$ by F2, because $A + B \supseteq A$ and $A + B \supseteq B$. Thus for arbitrary $A$ and $B$ we have $(A^* + B^*)^* \subseteq A^{**} \cap B^{**}$, hence

$$(A^* + B^*)^{**} \supseteq (A^{**} \cap B^{**})^* .$$

Now let $A$ and $B$ be closed cones for which $A^* + B^*$ is closed; then $A^* + B^* \supseteq (A \cap B)^*$ by F5. □

**F7.** If $C$ and $C'$ are closed cones, and if $C + C'$ is closed, we have $(C + C')^* = C^* \cap C'^*$.

**Proof.** F6 says $(C^* \cap C'^*)^* = C^{**} + C'^{**}$; apply F5 and * again. □

**F8.** Let $S$ be any set of indices and let $\tau$ be all the indices not in $S$. Also let $A_{S} = \{ a \mid a_{s} = 0$ for all $s \in S \}$. Then

$$A_{\tau}^{*} = A_{S}^{*} .$$

**Proof.** If $b_{s} = 0$ for all $s \notin S$ and $a_{s} = 0$ for all $s \in S$, obviously $a \cdot b = 0$; so $A_{\tau}^{*} \subseteq A_{S}^{*}$. If $b_{s} \neq 0$ for some $s \notin S$ and $a_{s} = 0$ for all $t \neq s$ and $a_{s} = -b_{s}$, then $a \in A_{S}$ and $a \cdot b < 0$; so $b \notin A_{S}^{*}$, hence $A_{\tau}^{*} \supseteq A_{S}^{*}$. □

### 9. Definite Proof of a Semidefinite Fact

Now we are almost ready to prove the result needed in the proof of Lemma 7.

Let $D$ be the set of real symmetric positive semidefinite matrices (called “spuds” henceforth for brevity), considered as vectors in $N$-dimensional space, where $N = (n + 1)n/2$. We use the inner product $A \cdot B = \text{tr} A^{T}B$; this is justified if we divide off-diagonal elements by $\sqrt{2}$.

For example, if $n = 3$ the correspondence between 6-dimensional vectors and $3 \times 3$ symmetric matrices is

$$(a, b, c, d, e, f) \leftrightarrow \begin{pmatrix} a & d/\sqrt{2} & e/\sqrt{2} \\ d/\sqrt{2} & b & f/\sqrt{2} \\ e/\sqrt{2} & f/\sqrt{2} & c \end{pmatrix}$$

preserving sum, scalar product, and dot product. A real symmetric matrix $\notin D$ that has a negative eigenvalue $\lambda$, and we can write

$$A = Q \text{diag}(\lambda, \lambda, \ldots, \lambda) Q^{T} \qquad (9.1)$$

for some orthogonal matrix $Q$. Thus $x^{T}Ax = \lambda$ for the unit vector $x = Q(1, 0, \ldots, 0)^{T}$; it follows easily that $D$ is a closed cone.
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The cone of spuds turns out to be self-dual:

**F9.** $D^* = D$.

**Proof.** If $A$ and $B$ are spuds then $A = X^T X$ and $B = Y^T Y$ and

$$A \cdot B = \text{tr} X^T X Y^T Y = \text{tr} X Y Y^T X^T = (Y X^T) \cdot (Y X^T) \geq 0;$$

hence $D \subseteq D^*$. (In fact, this argument shows that $A \cdot B = 0$ if and only if $AB = 0$, for any spuds $A$ and $B$, since $A = A^T$.) Conversely, assuming (9.1), let $B = Q \diag(1, 0, \ldots, 0) Q^T$; then $B$ is a spud, and

$$A \cdot B = \text{tr} A^T B = \text{tr} Q \diag(\lambda, 0, \ldots, 0) Q^T = \lambda < 0.$$

So $A$ is not in $D^*$; this proves $D \supseteq D^*$. □

Let $E$ be the set of all real symmetric matrices such that $E_{uv} = 0$ when $u \rightarrow v$ in a graph $G$; let $F$ be the set of all real symmetric matrices such that $F_{uv} = 0$ when $u = v$ or $u \not\rightarrow v$. The Fact stated in Section 7 is now equivalent in our new notation to

**Fact.** $(D \cap E)^* \subseteq D + F$.

**Proof.** We have $D + F = D^* + E^*$ by F8 and F9. The argument in the Appendix below proves that $D + F$ is closed. Hence $(D \cap E)^* = D + F$ by F6. □

10. Another Characterization

Remember $\vartheta$, $\vartheta_1$, $\vartheta_2$, and $\vartheta_3$? We are now going to introduce yet another function

$$\vartheta_4(G, w) = \max \left\{ \sum_v c(b_v) w_v \right\}$$

where $b$ is an orthogonal labeling of $\overline{G}$. (10.1)

**Lemma.** $\vartheta_3(G, w) \leq \vartheta_4(G, w)$.

**Proof.** Suppose $b$ is a normalized orthogonal labeling of $\overline{G}$ that achieves the maximum $\vartheta_3$; and suppose the vectors of this labeling have dimension $d$. Let

$$x_k = \sum_v b_v \sqrt{w_v}, \quad \text{for } 1 \leq k \leq d;$$

(10.2)
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Transactions on Information Theory 2, 3 (September 1956), 8-19.

Addendum

The original paper tacitly assumed that the sum $A + B$ of closed cones
is closed; but that assumption is false. For example, if $C$ is the “three-
dimensional ice cream cone” defined by $\sqrt{c_1^2 + c_2^2} \leq c_3$, and if $C' = \{(b_1, b_2, b_3)^T \mid b_1 \geq b_3\}$, we have $C = C^*$ and $C'^* = \{(t, 0, -t)^T \mid t \geq 0\}$
and $(0, 1, 0)^T \in (C \cap C')^* \setminus (C'^* + C'^*)$. That violates the conclusion of F6.

This counterexample was pointed out by Evan DeCorte in 2014, who also found a valid proof of the crucial Fact in Section 9: Suppose we have
a convergent sequence $A_n + B_n \to C$, where $A_n \in D$ and $B_n \in F$; we
shall prove that $C \in D + F$, hence $D + F$ is closed.

Let $\lambda_{n1} \geq \cdots \geq \lambda_{nN}$ be the eigenvalues of $B_n$. We have $\lambda_{n1} = \Lambda(B_n) \geq 0$, because $\lambda_{n1} + \cdots + \lambda_{nN} = \operatorname{tr} B_n = 0$. We also have $\Lambda(B_n) \leq \Lambda(A_n + B_n)$ by (6.2): furthermore $\Lambda(A_n + B_n) \leq M$ for some $M$ and all $n$,
because of the convergence. Hence $\lambda_{nN} \geq -MN$, and all eigenvalues
of $B_n$ are uniformly bounded. Therefore there is a subsequence in which
all eigenvalues $\lambda_n$ converge to limiting values $\lambda_i$. It follows that $B_n \to B \in F$ and $A_n \to C - B \in D$.

Further information can be found in A. Galtman, “Spectral characterizations
of the Lovász number and the Delsarte number of a graph,”

Problem P6 has been solved for $p = 1/2$ by Uriel Feige and Robert
Krauthgamer, “The probable value of the Lovász–Schrijver relaxations
for maximum independent set,” SIAM Journal on Computing 32 (2003),
345-370.