Table 666

TWENTY QUESTIONS (SEE EXERCISE 90)

1. The first question whose answer is A is:
(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

2. The next question with the same answer as this one is:
(A) 4 (B) 6 (C) 8 (D) 10 (E) 12

3. The only two consecutive questions with identical answers are questions:
(A) 15 and 16 (B) 16 and 17 (C) 17 and 18 (D) 18 and 19 (E) 19 and 20

4. The answer to this question is the same as the answers to questions:
(A) 10 and 13 (B) 14 and 16 (C) 7 and 20 (D) 1 and 15 (E) 8 and 12

5. The answer to question 14 is:
(A) B (B) E (C) C (D) A (E) D

6. The answer to this question is:
(A) A (B) B (C) C (D) D (E) none of those

7. An answer that appears most often is:
(A) A (B) B (C) C (D) D (E) E

8. Ignoring answers that appear equally often, the least common answer is:
(A) A (B) B (C) C (D) D (E) E

9. The sum of all question numbers whose answers are correct and the same as this one is:
(A) $\in [59\,..\,62]$ (B) $\in [52\,..\,55]$ (C) $\in [44\,..\,49]$ (D) $\in [61\,..\,67]$ (E) $\in [44\,..\,53]$

10. The answer to question 17 is:
(A) D (B) B (C) A (D) E (E) wrong

11. The number of questions whose answer is D is:
(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

12. The number of *other* questions with the same answer as this one is the same as the number of questions with answer:
(A) B (B) C (C) D (D) E (E) none of those

13. The number of questions whose answer is E is:
(A) 5 (B) 4 (C) 3 (D) 2 (E) 1

14. No answer appears exactly this many times:
(A) 2 (B) 3 (C) 4 (D) 5 (E) none of those

15. The set of odd-numbered questions with answer A is:
(A) $\{7\}$ (B) $\{9\}$ (C) not $\{11\}$ (D) $\{13\}$ (E) $\{15\}$

16. The answer to question 8 is the same as the answer to question:
(A) 3 (B) 2 (C) 13 (D) 18 (E) 20

17. The answer to question 10 is:
(A) C (B) D (C) B (D) A (E) correct

18. The number of prime-numbered questions whose answers are vowels is:
(A) prime (B) square (C) odd (D) even (E) zero

19. The last question whose answer is B is:
(A) 14 (B) 15 (C) 16 (D) 17 (E) 18

20. The maximum score that can be achieved on this test is:
(A) 18 (B) 19 (C) 20 (D) indeterminate
(E) achievable only by getting this question wrong

▸ **90.** [*M29*] (Donald R. Woods, 2000.) Find all ways to maximize the number of correct answers to the questionnaire in Table 666. Each question must be answered with a letter from A to E. *Hint:* Begin by clarifying the exact meaning of this exercise. What answers are best for the following two-question, two-letter "warmup problem"?

1. (A) Answer 2 is B. (B) Answer 1 is A.

2. (A) Answer 1 is correct. (B) Either answer 2 is wrong or answer 1 is A, but not both.

91. [*HM28*] Show that exercise 90 has a surprising, somewhat paradoxical answer if two changes are made to Table 666: 9(E) becomes '$\in [39\,..\,43]$'; 15(C) becomes '$\{11\}$'.

90. Suppose there are n questions, whose answers each lie in a given set S. A *student* supplies an answer list $\alpha = a_1 \ldots a_n$, with each $a_j \in S$; a *grader* supplies a Boolean vector $\beta = x_1 \ldots x_n$. There is a Boolean function $f_{js}(\alpha, \beta)$ for each $j \in \{1, \ldots, n\}$ and each $s \in S$. A graded answer list (α, β) is *valid* if and only if $F(\alpha, \beta)$ is true, where

$$F(\alpha,\beta) = F(a_1 \ldots a_n, x_1 \ldots x_n) = \bigwedge_{j=1}^{n} \bigwedge_{s \in S} \bigl([a_j = s] \;\Rightarrow\; x_j \equiv f_{js}(\alpha,\beta)\bigr).$$

The *maximum score* is the largest value of $x_1 + \cdots + x_n$ over all graded answer lists (α, β) that are valid. A *perfect score* is achieved if and only if $F(\alpha, 1 \ldots 1)$ holds.

Thus, in the warmup problem we have $n = 2$, $S = \{\mathrm{A}, \mathrm{B}\}$; $f_{1\mathrm{A}} = [a_2 = \mathrm{B}]$; $f_{1\mathrm{B}} = [a_1 = \mathrm{A}]$; $f_{2\mathrm{A}} = x_1$; $f_{2\mathrm{B}} = \bar{x}_2 \oplus [a_1 = \mathrm{A}]$. The four possible answer lists are:

$$\begin{array}{ll}
\mathrm{AA}: & F = (x_1 \equiv [\mathrm{A} = \mathrm{B}]) \wedge (x_2 \equiv x_1) \\
\mathrm{AB}: & F = (x_1 \equiv [\mathrm{B} = \mathrm{B}]) \wedge (x_2 \equiv \bar{x}_2 \oplus [\mathrm{A} = \mathrm{A}]) \\
\mathrm{BA}: & F = (x_1 \equiv [\mathrm{B} = \mathrm{A}]) \wedge (x_2 \equiv x_1) \\
\mathrm{BB}: & F = (x_1 \equiv [\mathrm{B} = \mathrm{A}]) \wedge (x_2 \equiv \bar{x}_2 \oplus [\mathrm{B} = \mathrm{A}])
\end{array}$$

Thus AA and BA must be graded 00; AB can be graded either 10 or 11; and BB has no valid grading. Only AB can achieve the maximum score, 2; but 2 isn't guaranteed.

⋮ (see `http://www-cs-faculty.stanford.edu/~knuth/fasc5b.ps.gz`)

A perfect score turns out to be impossible, but here are some well-graded answer lists:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
D	C	E	A	B	E	B	C	E	A	B	E	A	E	D	B	D	A	b	B	(i)
A	E	D	C	A	B	C	D	C	A	C	E	D	B	C	A	D	A	A	c	(ii)
D	C	E	A	B	A	D	C	D	A	E	D	A	E	D	B	D	B	E	e	(iii)
B	A	d	A	B	E	D	C	D	A	E	D	A	E	D	E	D	B	E	c	(iv)
A	E	D	C	A	B	C	D	C	A	C	E	D	B	a	C	D	A	A	c	(v)
D	e	D	A	B	E	D	e	C	A	E	D	A	E	D	B	D	C	C	A	(vi)
D	C	E	A	B	E	D	C	E	A	E	B	A	E	D	B	D	A	d	D	(vii)

(The incorrect answers are shown here as lowercase letters. Solutions (i)–(iii) are valid for exercise 90, while (iii)–(vii) are valid for exercise 91.)

91. Paradoxical situations can arise when the global function F of answer 90 is used recursively within one or more of the local functions f_{js}. Let's explore a bit of recursive territory by considering the following two-question, two-letter example:

1. (A) Answer 1 is incorrect. (B) Answer 2 is incorrect.
2. (A) Some answers can't be graded consistently. (B) No answers achieve a perfect score.

Here we have $f_{1\mathrm{A}} = \bar{x}_1$; $f_{1\mathrm{B}} = \bar{x}_2$; $f_{2\mathrm{A}} = \exists a_1 \exists a_2 \forall x_1 \forall x_2 \neg F(a_1 a_2, x_1 x_2)$; $f_{2\mathrm{B}} = \forall a_1 \forall a_2 \neg F(a_1 a_2, 11)$. (Formulas quantified by $\exists a$ or $\forall a$ expand into $|S|$ terms, while $\exists x$ or $\forall x$ expand into two; for example, $\exists a \forall x\, g(a, x) = \bigl(g(\mathrm{A},0) \wedge g(\mathrm{A},1)\bigr) \vee \bigl(g(\mathrm{B},0) \wedge g(\mathrm{B},1)\bigr)$ when $S = \{\mathrm{A}, \mathrm{B}\}$.) Sometimes the expansion is undefined, because it has more than one "fixed point"; but in this case there's no problem because $f_{2\mathrm{A}}$ is true: Answer AA can't be graded, since 1A implies $x_1 \equiv \bar{x}_1$. Also $f_{2\mathrm{B}}$ is true, because both BA and BB imply $x_1 \equiv \bar{x}_2$. Thus we get the maximum score 1 with either BA or BB and grades 01.

On the other hand the simple one-question, one-letter questionnaire '**1.** (A) The maximum score is 1' has an *indeterminate* maximum score. For in this case $f_{1\mathrm{A}} = F(\mathrm{A}, 1)$. We find that if $F(\mathrm{A}, 1) = 0$, only $(\mathrm{A}, 0)$ is a valid grading, so the only possible score is 0; similarly, if $F(\mathrm{A}, 1) = 1$, the only possible score is 1.

OK, suppose that the maximum score for the modified Table 666 is m. If m is determinate, an inconsistency arises; but if m is indeterminate, it's definitely 19.