| $p_k$ = pebbles in pit $k$ | | | | | | | | $n$ pebbles | $m_k$ = times pit $k$ is emptied | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $p_8$ | $p_7$ | $p_6$ | $p_5$ | $p_4$ | $p_3$ | $p_2$ | $p_1$ | | $m_8$ | $m_7$ | $m_6$ | $m_5$ | $m_4$ | $m_3$ | $m_2$ | $m_1$ |
| 8 | 6 | 4 | 2 | 4 | 0 | 0 | 0 | 24 | 1 | 1 | 1 | 1 | 2 | 2 | 4 | 12 |
| 8 | 6 | 4 | 2 | 0 | 1 | 1 | 1 | 23 | 1 | 1 | 1 | 1 | 1 | 2 | 4 | 12 |
| 8 | 6 | 4 | 2 | 0 | 1 | 1 | 0 | 22 | 1 | 1 | 1 | 1 | 1 | 2 | 4 | 11 |
| 0 | 7 | 5 | 3 | 1 | 2 | 2 | 1 | 21 | 0 | 1 | 1 | 1 | 1 | 2 | 4 | 11 |
| 0 | 7 | 5 | 3 | 1 | 2 | 2 | 0 | 20 | 0 | 1 | 1 | 1 | 1 | 2 | 4 | 10 |
| 0 | 7 | 5 | 3 | 1 | 2 | 0 | 1 | 19 | 0 | 1 | 1 | 1 | 1 | 2 | 3 | 10 |
| 0 | 7 | 5 | 3 | 1 | 2 | 0 | 0 | 18 | 0 | 1 | 1 | 1 | 1 | 2 | 3 | 9 |
| 0 | 0 | 6 | 4 | 2 | 3 | 1 | 1 | 17 | 0 | 0 | 1 | 1 | 1 | 2 | 3 | 9 |
| 0 | 0 | 6 | 4 | 2 | 3 | 1 | 0 | 16 | 0 | 0 | 1 | 1 | 1 | 2 | 3 | 8 |
| 0 | 0 | 6 | 4 | 2 | 0 | 2 | 1 | 15 | 0 | 0 | 1 | 1 | 1 | 1 | 3 | 8 |
| 0 | 0 | 6 | 4 | 2 | 0 | 2 | 0 | 14 | 0 | 0 | 1 | 1 | 1 | 1 | 3 | 7 |
| 0 | 0 | 6 | 4 | 2 | 0 | 0 | 1 | 13 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 7 |
| 0 | 0 | 6 | 4 | 2 | 0 | 0 | 0 | 12 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 6 |
| 0 | 0 | 0 | 5 | 3 | 1 | 1 | 1 | 11 | 0 | 0 | 0 | 1 | 1 | 1 | 2 | 6 |
| 0 | 0 | 0 | 5 | 3 | 1 | 1 | 0 | 10 | 0 | 0 | 0 | 1 | 1 | 1 | 2 | 5 |
| 0 | 0 | 0 | 0 | 4 | 2 | 2 | 1 | 9 | 0 | 0 | 0 | 0 | 1 | 1 | 2 | 5 |
| 0 | 0 | 0 | 0 | 4 | 2 | 2 | 0 | 8 | 0 | 0 | 0 | 0 | 1 | 1 | 2 | 4 |
| 0 | 0 | 0 | 0 | 4 | 2 | 0 | 1 | 7 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 4 |
| 0 | 0 | 0 | 0 | 4 | 2 | 0 | 0 | 6 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 3 |
| 0 | 0 | 0 | 0 | 0 | 3 | 1 | 1 | 5 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 3 |
| 0 | 0 | 0 | 0 | 0 | 3 | 1 | 0 | 4 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 2 |
| 0 | 0 | 0 | 0 | 0 | 0 | 2 | 1 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 |
| 0 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $p_k \in [0..k]$ | | | | | | | | | $p_k \leq k m_k$ | | | | | | | |

$q_k = p_{k+1} + p_{k+2} + \cdots$

| $q_7$ | $q_6$ | $q_5$ | $q_4$ | $q_3$ | $q_2$ | $q_1$ | $q_0$ |
|---|---|---|---|---|---|---|---|
| 8 | 14 | 18 | 20 | 24 | 24 | 24 | 24 |
| 8 | 14 | 18 | 20 | 20 | 21 | 22 | 23 |
| 8 | 14 | 18 | 20 | 20 | 21 | 22 | 22 |
| 0 | 7 | 12 | 15 | 16 | 18 | 20 | 21 |
| 0 | 7 | 12 | 15 | 16 | 18 | 20 | 20 |
| 0 | 7 | 12 | 15 | 16 | 18 | 18 | 19 |
| 0 | 7 | 12 | 15 | 16 | 18 | 18 | 18 |
| 0 | 0 | 6 | 10 | 12 | 15 | 16 | 17 |
| 0 | 0 | 6 | 10 | 12 | 15 | 16 | 16 |
| 0 | 0 | 6 | 10 | 12 | 12 | 14 | 15 |
| 0 | 0 | 6 | 10 | 12 | 12 | 14 | 14 |
| 0 | 0 | 6 | 10 | 12 | 12 | 12 | 13 |
| 0 | 0 | 6 | 10 | 12 | 12 | 12 | 12 |
| 0 | 0 | 0 | 5 | 8 | 9 | 10 | 11 |
| 0 | 0 | 0 | 5 | 8 | 9 | 10 | 10 |
| 0 | 0 | 0 | 0 | 4 | 6 | 8 | 9 |
| 0 | 0 | 0 | 0 | 4 | 6 | 8 | 8 |
| 0 | 0 | 0 | 0 | 4 | 6 | 6 | 7 |
| 0 | 0 | 0 | 0 | 4 | 6 | 6 | 6 |
| 0 | 0 | 0 | 0 | 0 | 3 | 4 | 5 |
| 0 | 0 | 0 | 0 | 0 | 3 | 4 | 4 |
| 0 | 0 | 0 | 0 | 0 | 0 | 2 | 3 |
| 0 | 0 | 0 | 0 | 0 | 0 | 2 | 2 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

$q_k + p_k = q_{k-1}$

$p_k$ = pebbles in pit $k$

| $p_8$ | $p_7$ | $p_6$ | $p_5$ | $p_4$ | $p_3$ | $p_2$ | $p_1$ |
|---|---|---|---|---|---|---|---|
| 8 | 6 | 4 | 2 | 4 | 0 | 0 | 0 |
| 8 | 6 | 4 | 2 | 0 | 1 | 1 | 1 |
| 8 | 6 | 4 | 2 | 0 | 1 | 1 | 0 |
| 0 | 7 | 5 | 3 | 1 | 2 | 2 | 1 |
| 0 | 7 | 5 | 3 | 1 | 2 | 2 | 0 |
| 0 | 7 | 5 | 3 | 1 | 2 | 0 | 1 |
| 0 | 7 | 5 | 3 | 1 | 2 | 0 | 0 |
| 0 | 0 | 6 | 4 | 2 | 3 | 1 | 1 |
| 0 | 0 | 6 | 4 | 2 | 3 | 1 | 0 |
| 0 | 0 | 6 | 4 | 2 | 0 | 2 | 1 |
| 0 | 0 | 6 | 4 | 2 | 0 | 2 | 0 |
| 0 | 0 | 6 | 4 | 2 | 0 | 0 | 1 |
| 0 | 0 | 6 | 4 | 2 | 0 | 0 | 0 |
| 0 | 0 | 0 | 5 | 3 | 1 | 1 | 1 |
| 0 | 0 | 0 | 5 | 3 | 1 | 1 | 0 |
| 0 | 0 | 0 | 0 | 4 | 2 | 2 | 1 |
| 0 | 0 | 0 | 0 | 4 | 2 | 2 | 0 |
| 0 | 0 | 0 | 0 | 4 | 2 | 0 | 1 |
| 0 | 0 | 0 | 0 | 4 | 2 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 3 | 1 | 1 |
| 0 | 0 | 0 | 0 | 0 | 3 | 1 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 2 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 2 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

$p_k \in [0 \,..\, k]$

$l_k = m_{k+1} + m_{k+2} + \cdots$

| $l_7$ | $l_6$ | $l_5$ | $l_4$ | $l_3$ | $l_2$ | $l_1$ | $l_0$ |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 6 | 8 | 12 | 24 |
| 1 | 2 | 3 | 4 | 5 | 7 | 11 | 23 |
| 1 | 2 | 3 | 4 | 5 | 7 | 11 | 22 |
| 0 | 1 | 2 | 3 | 4 | 6 | 10 | 21 |
| 0 | 1 | 2 | 3 | 4 | 6 | 10 | 20 |
| 0 | 1 | 2 | 3 | 4 | 6 | 9 | 19 |
| 0 | 1 | 2 | 3 | 4 | 6 | 9 | 18 |
| 0 | 0 | 1 | 2 | 3 | 5 | 8 | 17 |
| 0 | 0 | 1 | 2 | 3 | 5 | 8 | 16 |
| 0 | 0 | 1 | 2 | 3 | 4 | 7 | 15 |
| 0 | 0 | 1 | 2 | 3 | 4 | 7 | 14 |
| 0 | 0 | 1 | 2 | 3 | 4 | 6 | 13 |
| 0 | 0 | 1 | 2 | 3 | 4 | 6 | 12 |
| 0 | 0 | 0 | 1 | 2 | 3 | 5 | 11 |
| 0 | 0 | 0 | 1 | 2 | 3 | 5 | 10 |
| 0 | 0 | 0 | 0 | 1 | 2 | 4 | 9 |
| 0 | 0 | 0 | 0 | 1 | 2 | 4 | 8 |
| 0 | 0 | 0 | 0 | 1 | 2 | 3 | 7 |
| 0 | 0 | 0 | 0 | 1 | 2 | 3 | 6 |
| 0 | 0 | 0 | 0 | 0 | 1 | 2 | 5 |
| 0 | 0 | 0 | 0 | 0 | 1 | 2 | 4 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 3 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

$l_k + m_k = l_{k-1}$
$l_k + p_k = k m_k$

$m_k$ = times pit $k$ is emptied

| $m_8$ | $m_7$ | $m_6$ | $m_5$ | $m_4$ | $m_3$ | $m_2$ | $m_1$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 2 | 2 | 4 | 12 |
| 1 | 1 | 1 | 1 | 1 | 2 | 4 | 12 |
| 1 | 1 | 1 | 1 | 1 | 2 | 4 | 11 |
| 0 | 1 | 1 | 1 | 1 | 2 | 4 | 11 |
| 0 | 1 | 1 | 1 | 1 | 2 | 4 | 10 |
| 0 | 1 | 1 | 1 | 1 | 2 | 3 | 10 |
| 0 | 1 | 1 | 1 | 1 | 2 | 3 | 9 |
| 0 | 0 | 1 | 1 | 1 | 2 | 3 | 9 |
| 0 | 0 | 1 | 1 | 1 | 2 | 3 | 8 |
| 0 | 0 | 1 | 1 | 1 | 1 | 3 | 8 |
| 0 | 0 | 1 | 1 | 1 | 1 | 3 | 7 |
| 0 | 0 | 1 | 1 | 1 | 1 | 2 | 7 |
| 0 | 0 | 1 | 1 | 1 | 1 | 2 | 6 |
| 0 | 0 | 0 | 1 | 1 | 1 | 2 | 6 |
| 0 | 0 | 0 | 1 | 1 | 1 | 2 | 5 |
| 0 | 0 | 0 | 0 | 1 | 1 | 2 | 5 |
| 0 | 0 | 0 | 0 | 1 | 1 | 2 | 4 |
| 0 | 0 | 0 | 0 | 1 | 1 | 1 | 4 |
| 0 | 0 | 0 | 0 | 1 | 1 | 1 | 3 |
| 0 | 0 | 0 | 0 | 0 | 1 | 1 | 3 |
| 0 | 0 | 0 | 0 | 0 | 1 | 1 | 2 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

$q_k = p_{k+1} + p_{k+2} + \cdots$

| $q_7$ | $q_6$ | $q_5$ | $q_4$ | $q_3$ | $q_2$ | $q_1$ | $q_0$ |
|---|---|---|---|---|---|---|---|
| 8 | 14 | 18 | 20 | 24 | 24 | 24 | 24 |
| 8 | 14 | 18 | 20 | 20 | 21 | 22 | 23 |
| 8 | 14 | 18 | 20 | 20 | 21 | 22 | 22 |
| 0 | 7 | 12 | 15 | 16 | 18 | 20 | 21 |
| 0 | 7 | 12 | 15 | 16 | 18 | 20 | 20 |
| 0 | 7 | 12 | 15 | 16 | 18 | 18 | 19 |
| 0 | 7 | 12 | 15 | 16 | 18 | 18 | 18 |
| 0 | 0 | 6 | 10 | 12 | 15 | 16 | 17 |
| 0 | 0 | 6 | 10 | 12 | 15 | 16 | 16 |
| 0 | 0 | 6 | 10 | 12 | 12 | 14 | 15 |
| 0 | 0 | 6 | 10 | 12 | 12 | 14 | 14 |
| 0 | 0 | 6 | 10 | 12 | 12 | 12 | 13 |
| 0 | 0 | 6 | 10 | 12 | 12 | 12 | 12 |
| 0 | 0 | 0 | 5 | 8 | 9 | 10 | 11 |
| 0 | 0 | 0 | 5 | 8 | 9 | 10 | 10 |
| 0 | 0 | 0 | 0 | 4 | 6 | 8 | 9 |
| 0 | 0 | 0 | 0 | 4 | 6 | 8 | 8 |
| 0 | 0 | 0 | 0 | 4 | 6 | 6 | 7 |
| 0 | 0 | 0 | 0 | 4 | 6 | 6 | 6 |
| 0 | 0 | 0 | 0 | 0 | 3 | 4 | 5 |
| 0 | 0 | 0 | 0 | 0 | 3 | 4 | 4 |
| 0 | 0 | 0 | 0 | 0 | 0 | 2 | 3 |
| 0 | 0 | 0 | 0 | 0 | 0 | 2 | 2 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

$q_k + p_k = q_{k-1}$
$q_k = (k+1)l_k$

$p_k$ = pebbles in pit $k$

| $p_8$ | $p_7$ | $p_6$ | $p_5$ | $p_4$ | $p_3$ | $p_2$ | $p_1$ |
|---|---|---|---|---|---|---|---|
| 8 | 6 | 4 | 2 | 4 | 0 | 0 | 0 |
| 8 | 6 | 4 | 2 | 0 | 1 | 1 | 1 |
| 8 | 6 | 4 | 2 | 0 | 1 | 1 | 0 |
| 0 | 7 | 5 | 3 | 1 | 2 | 2 | 1 |
| 0 | 7 | 5 | 3 | 1 | 2 | 2 | 0 |
| 0 | 7 | 5 | 3 | 1 | 2 | 0 | 1 |
| 0 | 7 | 5 | 3 | 1 | 2 | 0 | 0 |
| 0 | 0 | 6 | 4 | 2 | 3 | 1 | 1 |
| 0 | 0 | 6 | 4 | 2 | 3 | 1 | 0 |
| 0 | 0 | 6 | 4 | 2 | 0 | 2 | 1 |
| 0 | 0 | 6 | 4 | 2 | 0 | 2 | 0 |
| 0 | 0 | 6 | 4 | 2 | 0 | 0 | 1 |
| 0 | 0 | 6 | 4 | 2 | 0 | 0 | 0 |
| 0 | 0 | 0 | 5 | 3 | 1 | 1 | 1 |
| 0 | 0 | 0 | 5 | 3 | 1 | 1 | 0 |
| 0 | 0 | 0 | 0 | 4 | 2 | 2 | 1 |
| 0 | 0 | 0 | 0 | 4 | 2 | 2 | 0 |
| 0 | 0 | 0 | 0 | 4 | 2 | 0 | 1 |
| 0 | 0 | 0 | 0 | 4 | 2 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 3 | 1 | 1 |
| 0 | 0 | 0 | 0 | 0 | 3 | 1 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 2 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 2 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

$p_k \in [0\,..\,k]$
$p_k = q_{k-1} \bmod (k+1)$

$l_k = m_{k+1} + m_{k+2} + \cdots$

| $l_7$ | $l_6$ | $l_5$ | $l_4$ | $l_3$ | $l_2$ | $l_1$ | $l_0$ |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 6 | 8 | 12 | 24 |
| 1 | 2 | 3 | 4 | 5 | 7 | 11 | 23 |
| 1 | 2 | 3 | 4 | 5 | 7 | 11 | 22 |
| 0 | 1 | 2 | 3 | 4 | 6 | 10 | 21 |
| 0 | 1 | 2 | 3 | 4 | 6 | 10 | 20 |
| 0 | 1 | 2 | 3 | 4 | 6 | 9 | 19 |
| 0 | 1 | 2 | 3 | 4 | 6 | 9 | 18 |
| 0 | 0 | 1 | 2 | 3 | 5 | 8 | 17 |
| 0 | 0 | 1 | 2 | 3 | 5 | 8 | 16 |
| 0 | 0 | 1 | 2 | 3 | 4 | 7 | 15 |
| 0 | 0 | 1 | 2 | 3 | 4 | 7 | 14 |
| 0 | 0 | 1 | 2 | 3 | 4 | 6 | 13 |
| 0 | 0 | 1 | 2 | 3 | 4 | 6 | 12 |
| 0 | 0 | 0 | 1 | 2 | 3 | 5 | 11 |
| 0 | 0 | 0 | 1 | 2 | 3 | 5 | 10 |
| 0 | 0 | 0 | 0 | 1 | 2 | 4 | 9 |
| 0 | 0 | 0 | 0 | 1 | 2 | 4 | 8 |
| 0 | 0 | 0 | 0 | 1 | 2 | 3 | 7 |
| 0 | 0 | 0 | 0 | 1 | 2 | 3 | 6 |
| 0 | 0 | 0 | 0 | 0 | 1 | 2 | 5 |
| 0 | 0 | 0 | 0 | 0 | 1 | 2 | 4 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 3 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

$l_k + m_k = l_{k-1}$
$l_k + p_k = km_k$

$m_k$ = times pit $k$ is emptied

| $m_8$ | $m_7$ | $m_6$ | $m_5$ | $m_4$ | $m_3$ | $m_2$ | $m_1$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 2 | 2 | 4 | 12 |
| 1 | 1 | 1 | 1 | 1 | 2 | 4 | 12 |
| 1 | 1 | 1 | 1 | 1 | 2 | 4 | 11 |
| 0 | 1 | 1 | 1 | 1 | 2 | 4 | 11 |
| 0 | 1 | 1 | 1 | 1 | 2 | 4 | 10 |
| 0 | 1 | 1 | 1 | 1 | 2 | 3 | 10 |
| 0 | 1 | 1 | 1 | 1 | 2 | 3 | 9 |
| 0 | 0 | 1 | 1 | 1 | 2 | 3 | 9 |
| 0 | 0 | 1 | 1 | 1 | 2 | 3 | 8 |
| 0 | 0 | 1 | 1 | 1 | 1 | 3 | 8 |
| 0 | 0 | 1 | 1 | 1 | 1 | 3 | 7 |
| 0 | 0 | 1 | 1 | 1 | 1 | 2 | 7 |
| 0 | 0 | 1 | 1 | 1 | 1 | 2 | 6 |
| 0 | 0 | 0 | 1 | 1 | 1 | 2 | 6 |
| 0 | 0 | 0 | 1 | 1 | 1 | 2 | 5 |
| 0 | 0 | 0 | 0 | 1 | 1 | 2 | 5 |
| 0 | 0 | 0 | 0 | 1 | 1 | 2 | 4 |
| 0 | 0 | 0 | 0 | 1 | 1 | 1 | 4 |
| 0 | 0 | 0 | 0 | 1 | 1 | 1 | 3 |
| 0 | 0 | 0 | 0 | 0 | 1 | 1 | 3 |
| 0 | 0 | 0 | 0 | 0 | 1 | 1 | 2 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

$p_{k+1} - p_k = (k+2)m_{k+1} - km_k$

$q_k = p_{k+1} + p_{k+2} + \cdots$

| $q_7$ | $q_6$ | $q_5$ | $q_4$ | $q_3$ | $q_2$ | $q_1$ | $q_0$ |
|---|---|---|---|---|---|---|---|
| 8 | 14 | 18 | 20 | 24 | 24 | 24 | 24 |
| 8 | 14 | 18 | 20 | 20 | 21 | 22 | 23 |
| 8 | 14 | 18 | 20 | 20 | 21 | 22 | 22 |
| 0 | 7 | 12 | 15 | 16 | 18 | 20 | 21 |
| 0 | 7 | 12 | 15 | 16 | 18 | 20 | 20 |
| 0 | 7 | 12 | 15 | 16 | 18 | 18 | 19 |
| 0 | 7 | 12 | 15 | 16 | 18 | 18 | 18 |
| 0 | 0 | 6 | 10 | 12 | 15 | 16 | 17 |
| 0 | 0 | 6 | 10 | 12 | 15 | 16 | 16 |
| 0 | 0 | 6 | 10 | 12 | 12 | 14 | 15 |
| 0 | 0 | 6 | 10 | 12 | 12 | 14 | 14 |
| 0 | 0 | 6 | 10 | 12 | 12 | 12 | 13 |
| 0 | 0 | 6 | 10 | 12 | 12 | 12 | 12 |
| 0 | 0 | 0 | 5 | 8 | 9 | 10 | 11 |
| 0 | 0 | 0 | 5 | 8 | 9 | 10 | 10 |
| 0 | 0 | 0 | 0 | 4 | 6 | 8 | 9 |
| 0 | 0 | 0 | 0 | 4 | 6 | 8 | 8 |
| 0 | 0 | 0 | 0 | 4 | 6 | 6 | 7 |
| 0 | 0 | 0 | 0 | 4 | 6 | 6 | 6 |
| 0 | 0 | 0 | 0 | 0 | 3 | 4 | 5 |
| 0 | 0 | 0 | 0 | 0 | 3 | 4 | 4 |
| 0 | 0 | 0 | 0 | 0 | 0 | 2 | 3 |
| 0 | 0 | 0 | 0 | 0 | 0 | 2 | 2 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

$q_k + p_k = q_{k-1}$
$q_k = (k+1)l_k$

$p_k$ = pebbles in pit $k$

| $p_8$ | $p_7$ | $p_6$ | $p_5$ | $p_4$ | $p_3$ | $p_2$ | $p_1$ |
|---|---|---|---|---|---|---|---|
| 8 | 6 | 4 | 2 | 4 | 0 | 0 | 0 |
| 8 | 6 | 4 | 2 | 0 | 1 | 1 | 1 |
| 8 | 6 | 4 | 2 | 0 | 1 | 1 | 0 |
| 0 | 7 | 5 | 3 | 1 | 2 | 2 | 1 |
| 0 | 7 | 5 | 3 | 1 | 2 | 2 | 0 |
| 0 | 7 | 5 | 3 | 1 | 2 | 0 | 1 |
| 0 | 7 | 5 | 3 | 1 | 2 | 0 | 0 |
| 0 | 0 | 6 | 4 | 2 | 3 | 1 | 1 |
| 0 | 0 | 6 | 4 | 2 | 3 | 1 | 0 |
| 0 | 0 | 6 | 4 | 2 | 0 | 2 | 1 |
| 0 | 0 | 6 | 4 | 2 | 0 | 2 | 0 |
| 0 | 0 | 6 | 4 | 2 | 0 | 0 | 1 |
| 0 | 0 | 6 | 4 | 2 | 0 | 0 | 0 |
| 0 | 0 | 0 | 5 | 3 | 1 | 1 | 1 |
| 0 | 0 | 0 | 5 | 3 | 1 | 1 | 0 |
| 0 | 0 | 0 | 0 | 4 | 2 | 2 | 1 |
| 0 | 0 | 0 | 0 | 4 | 2 | 2 | 0 |
| 0 | 0 | 0 | 0 | 4 | 2 | 0 | 1 |
| 0 | 0 | 0 | 0 | 4 | 2 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 3 | 1 | 1 |
| 0 | 0 | 0 | 0 | 0 | 3 | 1 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 2 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 2 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

$p_k \in [0\,..\,k]$
$p_k = q_{k-1} \bmod (k+1)$

$l_k = m_{k+1} + m_{k+2} + \cdots$

| $l_7$ | $l_6$ | $l_5$ | $l_4$ | $l_3$ | $l_2$ | $l_1$ | $l_0$ |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 6 | 8 | 12 | 24 |
| 1 | 2 | 3 | 4 | 5 | 7 | 11 | 23 |
| 1 | 2 | 3 | 4 | 5 | 7 | 11 | 22 |
| 0 | 1 | 2 | 3 | 4 | 6 | 10 | 21 |
| 0 | 1 | 2 | 3 | 4 | 6 | 10 | 20 |
| 0 | 1 | 2 | 3 | 4 | 6 | 9 | 19 |
| 0 | 1 | 2 | 3 | 4 | 6 | 9 | 18 |
| 0 | 0 | 1 | 2 | 3 | 5 | 8 | 17 |
| 0 | 0 | 1 | 2 | 3 | 5 | 8 | 16 |
| 0 | 0 | 1 | 2 | 3 | 4 | 7 | 15 |
| 0 | 0 | 1 | 2 | 3 | 4 | 7 | 14 |
| 0 | 0 | 1 | 2 | 3 | 4 | 6 | 13 |
| 0 | 0 | 1 | 2 | 3 | 4 | 6 | 12 |
| 0 | 0 | 0 | 1 | 2 | 3 | 5 | 11 |
| 0 | 0 | 0 | 1 | 2 | 3 | 5 | 10 |
| 0 | 0 | 0 | 0 | 1 | 2 | 4 | 9 |
| 0 | 0 | 0 | 0 | 1 | 2 | 4 | 8 |
| 0 | 0 | 0 | 0 | 1 | 2 | 3 | 7 |
| 0 | 0 | 0 | 0 | 1 | 2 | 3 | 6 |
| 0 | 0 | 0 | 0 | 0 | 1 | 2 | 5 |
| 0 | 0 | 0 | 0 | 0 | 1 | 2 | 4 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 3 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

$l_k + m_k = l_{k-1}$
$l_k + p_k = km_k$
$l_k = \lfloor \frac{k}{k+1} l_{k-1} \rfloor$

$m_k$ = times pit $k$ is emptied

| $m_8$ | $m_7$ | $m_6$ | $m_5$ | $m_4$ | $m_3$ | $m_2$ | $m_1$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 2 | 2 | 4 | 12 |
| 1 | 1 | 1 | 1 | 1 | 2 | 4 | 12 |
| 1 | 1 | 1 | 1 | 1 | 2 | 4 | 11 |
| 0 | 1 | 1 | 1 | 1 | 2 | 4 | 11 |
| 0 | 1 | 1 | 1 | 1 | 2 | 4 | 10 |
| 0 | 1 | 1 | 1 | 1 | 2 | 3 | 10 |
| 0 | 1 | 1 | 1 | 1 | 2 | 3 | 9 |
| 0 | 0 | 1 | 1 | 1 | 2 | 3 | 9 |
| 0 | 0 | 1 | 1 | 1 | 2 | 3 | 8 |
| 0 | 0 | 1 | 1 | 1 | 1 | 3 | 8 |
| 0 | 0 | 1 | 1 | 1 | 1 | 3 | 7 |
| 0 | 0 | 1 | 1 | 1 | 1 | 2 | 7 |
| 0 | 0 | 1 | 1 | 1 | 1 | 2 | 6 |
| 0 | 0 | 0 | 1 | 1 | 1 | 2 | 6 |
| 0 | 0 | 0 | 1 | 1 | 1 | 2 | 5 |
| 0 | 0 | 0 | 0 | 1 | 1 | 2 | 5 |
| 0 | 0 | 0 | 0 | 1 | 1 | 2 | 4 |
| 0 | 0 | 0 | 0 | 1 | 1 | 1 | 4 |
| 0 | 0 | 0 | 0 | 1 | 1 | 1 | 3 |
| 0 | 0 | 0 | 0 | 0 | 1 | 1 | 3 |
| 0 | 0 | 0 | 0 | 0 | 1 | 1 | 2 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

$p_{k+1} - p_k = (k+2)m_{k+1} - km_k$
$m_k = \lceil l_{k-1}/(k+1) \rceil$

Key facts:

$$l_0 = n$$
$$l_k = \lfloor \frac{k}{k+1} l_{k-1} \rfloor$$
$$q_k = (k+1) l_k$$
$$p_k = q_{k-1} \bmod (k+1)$$

Example, $n = 83$:

$l_0 = 83$; $q_0 = 83$; $p_1 = q_0 \bmod 2 = 1$.
$l_1 = \lfloor \frac{1}{2} \cdot 83 \rfloor = 41$; $q_1 = 2 \cdot 41 = 82$; $p_2 = q_1 \bmod 3 = 1$.
$l_2 = \lfloor \frac{2}{3} \cdot 41 \rfloor = 27$; $q_2 = 3 \cdot 27 = 81$; $p_3 = q_2 \bmod 4 = 1$.
$l_3 = \lfloor \frac{3}{4} \cdot 27 \rfloor = 20$; $q_3 = 4 \cdot 20 = 80$; $p_4 = q_3 \bmod 5 = 0$.
$l_4 = \lfloor \frac{4}{5} \cdot 20 \rfloor = 16$; $q_4 = 5 \cdot 16 = 80$; $p_5 = q_4 \bmod 6 = 2$.
$l_5 = \lfloor \frac{5}{6} \cdot 16 \rfloor = 13$; $q_5 = 6 \cdot 13 = 78$; $p_6 = q_5 \bmod 7 = 1$.
$l_6 = \lfloor \frac{6}{7} \cdot 13 \rfloor = 11$; $q_6 = 7 \cdot 11 = 77$; $p_7 = q_6 \bmod 8 = 5$.
$l_7 = \lfloor \frac{7}{8} \cdot 11 \rfloor = 9$; $q_7 = 8 \cdot 9 = 72$; $p_8 = q_7 \bmod 9 = 0$.
$l_8 = \lfloor \frac{8}{9} \cdot 9 \rfloor = 8$; $q_8 = 9 \cdot 8 = 72$; $p_9 = q_8 \bmod 10 = 2$.
$l_9 = \lfloor \frac{9}{10} \cdot 8 \rfloor = 7$; $q_9 = 10 \cdot 7 = 70$; $p_{10} = q_9 \bmod 11 = 4$.
$l_{10} = \lfloor \frac{10}{11} \cdot 7 \rfloor = 6$; $q_{10} = 11 \cdot 6 = 66$; $p_{11} = q_{10} \bmod 12 = 6$.
$l_{11} = \lfloor \frac{11}{12} \cdot 6 \rfloor = 5$; $q_{11} = 12 \cdot 5 = 60$; $p_{12} = q_{11} \bmod 13 = 8$.
$l_{12} = \lfloor \frac{12}{13} \cdot 5 \rfloor = 4$; $q_{12} = 13 \cdot 4 = 52$; $p_{13} = q_{12} \bmod 14 = 10$.
$l_{13} = \lfloor \frac{13}{14} \cdot 4 \rfloor = 3$; $q_{13} = 14 \cdot 3 = 42$; $p_{14} = q_{13} \bmod 15 = 12$.
$l_{14} = \lfloor \frac{14}{15} \cdot 3 \rfloor = 2$; $q_{14} = 15 \cdot 2 = 30$; $p_{15} = q_{14} \bmod 16 = 14$.
$l_{15} = \lfloor \frac{15}{16} \cdot 2 \rfloor = 1$; $q_{15} = 16 \cdot 1 = 16$; $p_{16} = q_{15} \bmod 17 = 16$.
$p_{16} p_{15} \ldots p_1 = 16\ 14\ 12\ 10\ 8\ 6\ 4\ 2\ 0\ 5\ 1\ 2\ 0\ 1\ 1\ 1$.

Example, $n = 84$:

$l_0 = 84$; $q_0 = 84$; $p_1 = q_0 \bmod 2 = 0$.
$l_1 = \lfloor \frac{1}{2} \cdot 84 \rfloor = 42$; $q_1 = 2 \cdot 42 = 84$; $p_2 = q_1 \bmod 3 = 0$.
$l_2 = \lfloor \frac{2}{3} \cdot 42 \rfloor = 28$; $q_2 = 3 \cdot 28 = 84$; $p_3 = q_2 \bmod 4 = 0$.
$l_3 = \lfloor \frac{3}{4} \cdot 28 \rfloor = 21$; $q_3 = 4 \cdot 21 = 84$; $p_4 = q_3 \bmod 5 = 4$.
$l_4 = \lfloor \frac{4}{5} \cdot 21 \rfloor = 16$; $q_4 = 5 \cdot 16 = 80$; $p_5 = q_4 \bmod 6 = 2$.
$l_5 = \lfloor \frac{5}{6} \cdot 16 \rfloor = 13$; $q_5 = 6 \cdot 13 = 78$; $p_6 = q_5 \bmod 7 = 1$.
$l_6 = \lfloor \frac{6}{7} \cdot 13 \rfloor = 11$; $q_6 = 7 \cdot 11 = 77$; $p_7 = q_6 \bmod 8 = 5$.
$l_7 = \lfloor \frac{7}{8} \cdot 11 \rfloor = 9$; $q_7 = 8 \cdot 9 = 72$; $p_8 = q_7 \bmod 9 = 0$.
$l_8 = \lfloor \frac{8}{9} \cdot 9 \rfloor = 8$; $q_8 = 9 \cdot 8 = 72$; $p_9 = q_8 \bmod 10 = 2$.
$l_9 = \lfloor \frac{9}{10} \cdot 8 \rfloor = 7$; $q_9 = 10 \cdot 7 = 70$; $p_{10} = q_9 \bmod 11 = 4$.
$l_{10} = \lfloor \frac{10}{11} \cdot 7 \rfloor = 6$; $q_{10} = 11 \cdot 6 = 66$; $p_{11} = q_{10} \bmod 12 = 6$.
$l_{11} = \lfloor \frac{11}{12} \cdot 6 \rfloor = 5$; $q_{11} = 12 \cdot 5 = 60$; $p_{12} = q_{11} \bmod 13 = 8$.
$l_{12} = \lfloor \frac{12}{13} \cdot 5 \rfloor = 4$; $q_{12} = 13 \cdot 4 = 52$; $p_{13} = q_{12} \bmod 14 = 10$.
$l_{13} = \lfloor \frac{13}{14} \cdot 4 \rfloor = 3$; $q_{13} = 14 \cdot 3 = 42$; $p_{14} = q_{13} \bmod 15 = 12$.
$l_{14} = \lfloor \frac{14}{15} \cdot 3 \rfloor = 2$; $q_{14} = 15 \cdot 2 = 30$; $p_{15} = q_{14} \bmod 16 = 14$.
$l_{15} = \lfloor \frac{15}{16} \cdot 2 \rfloor = 1$; $q_{15} = 16 \cdot 1 = 16$; $p_{16} = q_{15} \bmod 17 = 16$.
$p_{16} p_{15} \ldots p_1 = 16\ 14\ 12\ 10\ 8\ 6\ 4\ 2\ 0\ 5\ 1\ 2\ 4\ 0\ 0\ 0$.

$$\text{Ч}_n = \text{fewest pebbles that need pit } n.$$

$$\text{Ч}_1, \text{Ч}_2, \text{Ч}_3, \ldots \; = \; 1, 2, 4, 6, 10, 12, 18, 22, 30, 34, 42, 48, 58, 60, 78, 82, 102, 108, \ldots \; . \qquad \text{(N377, M1009, A2491)}$$

$$\text{Ч}_n = \left\lceil \frac{2}{1} \left\lceil \frac{3}{2} \left\lceil \frac{4}{3} \cdots \left\lceil \frac{n}{n-1} \right\rceil \cdots \right\rceil \right\rceil \right\rceil \qquad \text{(with } n-1 \text{ ceiling brackets).}$$

Example for $n = 10$: $\lceil \frac{10}{9} \rceil = 2$; $\lceil \frac{9}{8} \cdot 2 \rceil = 3$; $\lceil \frac{8}{7} \cdot 3 \rceil = 4$; $\lceil \frac{7}{6} \cdot 4 \rceil = 5$; $\lceil \frac{6}{5} \cdot 5 \rceil = 6$; $\lceil \frac{5}{4} \cdot 6 \rceil = 8$; $\lceil \frac{4}{3} \cdot 8 \rceil = 11$; $\lceil \frac{3}{2} \cdot 11 \rceil = 17$; $\lceil \frac{2}{1} \cdot 17 \rceil = 34$.

$$\text{Ч}_n \; = \; \frac{n^2}{\pi} + O(n^{4/3}). \qquad \text{(Erdős and Jabotinsky, 1958)}$$

For example, $\text{Ч}_{1000000} = 318310503562$, and $\lfloor 1000000000000/\pi \rfloor = 318309886183$.

*It is a pleasant surprise to see π arise from such a simple game.*
— NEIL J. A. SLOANE, *My favorite integer sequences* (1998)

A fairly elementary proof of this asymptotic formula by Broline and Loeb (1995) used the interesting sequence of fractions

$$\phi_m = \frac{1}{4^m}\binom{2m}{m} \; = \; \frac{2m-1}{2m}\frac{2m-3}{2m-2}\cdots\frac{1}{2},$$

which satisfy not only $\phi_m = \frac{2m-1}{2m}\phi_{m-1}$ but also

$$\phi_m = \frac{\phi_0 + \phi_1 + \cdots + \phi_{m-1}}{2m}.$$

(For example, $\bigl(1 + \frac{1}{2} + \frac{3}{8} + \frac{5}{16}\bigr)/8 = \frac{35}{128} = \frac{7}{8}\frac{5}{6}\frac{3}{4}\frac{1}{2}$.)

The sieve of Tchoukaillon

| | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| **1** | 2 | ~~3~~ | 4 | ~~5~~ | 6 | ~~7~~ | 8 | ~~9~~ | 10 | ~~11~~ | 12 | ~~13~~ | 14 | ~~15~~ | 16 | ~~17~~ | 18 | ~~19~~ | 20 | ~~21~~ | 22 | ~~23~~ | 24 | ~~25~~ | 26 | ~~27~~ | 28 | ~~29~~ | 30 | ~~31~~ | 32 | ~~33~~ | 34 | ~~35~~ | 36 | ~~37~~ | 38 | ~~39~~ | 40 |
| **1** | **2** | | 4 | | 6 | | ~~8~~ | | 10 | | 12 | | ~~14~~ | | 16 | | 18 | | ~~20~~ | | 22 | | 24 | | ~~26~~ | | 28 | | 30 | | ~~32~~ | | 34 | | 36 | | ~~38~~ | | 40 |
| **1** | **2** | | **4** | | 6 | | | | 10 | | 12 | | | | ~~16~~ | | 18 | | | | 22 | | 24 | | | | ~~28~~ | | 30 | | | | 34 | | 36 | | | | ~~40~~ |
| **1** | **2** | | **4** | | **6** | | | | 10 | | 12 | | | | | | 18 | | | | 22 | | ~~24~~ | | | | | | 30 | | | | 34 | | 36 | | | | |
| **1** | **2** | | **4** | | **6** | | | | **10** | | 12 | | | | | | 18 | | | | 22 | | | | | | | | 30 | | | | 34 | | ~~36~~ | | | | |
| **1** | **2** | | **4** | | **6** | | | | **10** | | **12** | | | | | | 18 | | | | 22 | | | | | | | | 30 | | | | 34 | | | | | | |
| **1** | **2** | | **4** | | **6** | | | | **10** | | **12** | | | | | | **18** | | | | 22 | | | | | | | | 30 | | | | 34 | | | | | | |
| **1** | **2** | | **4** | | **6** | | | | **10** | | **12** | | | | | | **18** | | | | **22** | | | | | | | | 30 | | | | 34 | | | | | | |
| **1** | **2** | | **4** | | **6** | | | | **10** | | **12** | | | | | | **18** | | | | **22** | | | | | | | | **30** | | | | 34 | | | | | | |
| **1** | **2** | | **4** | | **6** | | | | **10** | | **12** | | | | | | **18** | | | | **22** | | | | | | | | **30** | | | | **34** | | | | | | |

The Tchoukaillon arrays

$$\text{ч}^{(1)}_{q,1} = q+1; \quad \text{ч}^{(n+1)}_{qn+r,j} = \begin{cases} \text{ч}^{(n)}_{q(n+1)+r,j}, & \text{if } j+r<n; \\ \text{ч}^{(n)}_{q(n+1)+r+1,j-1}, & \text{if } j+r \geq n; \end{cases} \quad \text{for } n \geq 1,\ q \geq 0,\ 0 \leq r < q.$$

| $\text{ч}^{(1)}$ |
|---|
| 1 |
| 2 |
| 3 |
| 4 |
| 5 |
| 6 |
| ⋮ |

| $\text{ч}^{(2)}$ | |
|---|---|
| 1 | 2 |
| 3 | 4 |
| 5 | 6 |
| 7 | 8 |
| 9 | 10 |
| 11 | 12 |
| ⋮ | ⋮ |

| $\text{ч}^{(3)}$ | | |
|---|---|---|
| 1 | 2 | 4 |
| 3 | 5 | 6 |
| 7 | 8 | 10 |
| 9 | 11 | 12 |
| 13 | 14 | 16 |
| 15 | 17 | 18 |
| ⋮ | ⋮ | ⋮ |

| $\text{ч}^{(4)}$ | | | |
|---|---|---|---|
| 1 | 2 | 4 | 6 |
| 3 | 5 | 8 | 10 |
| 7 | 9 | 11 | 12 |
| 13 | 14 | 16 | 18 |
| 15 | 17 | 20 | 22 |
| 19 | 21 | 23 | 24 |
| ⋮ | ⋮ | ⋮ | ⋮ |

| $\text{ч}^{(5)}$ | | | | |
|---|---|---|---|---|
| 1 | 2 | 4 | 6 | 10 |
| 3 | 5 | 8 | 11 | 12 |
| 7 | 9 | 14 | 16 | 18 |
| 13 | 15 | 17 | 20 | 22 |
| 19 | 21 | 23 | 24 | 30 |
| 25 | 26 | 28 | 32 | 34 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |

| $\text{ч}^{(6)}$ | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 4 | 6 | 10 | 12 |
| 3 | 5 | 8 | 11 | 16 | 18 |
| 7 | 9 | 14 | 17 | 20 | 22 |
| 13 | 15 | 21 | 23 | 24 | 30 |
| 19 | 25 | 26 | 28 | 32 | 34 |
| 27 | 29 | 33 | 35 | 36 | 42 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |

| $\text{ч}^{(7)}$ | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 4 | 6 | 10 | 12 | 18 |
| 3 | 5 | 8 | 11 | 16 | 20 | 22 |
| 7 | 9 | 14 | 17 | 23 | 24 | 30 |
| 13 | 15 | 21 | 26 | 28 | 32 | 34 |
| 19 | 25 | 29 | 33 | 35 | 36 | 42 |
| 27 | 31 | 37 | 38 | 40 | 46 | 48 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |

The rows and columns of $\text{ч}^{(n)}$ are monotonically increasing. Furthermore, we have monotonicity in the “third dimension”:

$$\text{ч}^{(n)}_{i,j} \leq \text{ч}^{(n+1)}_{i,j}, \qquad \text{for all integers } i \geq 0,\ j \geq 0, \text{ and } n > j.$$

The first 32 rows and first 32 columns of $\text{ч}^{(\infty)}$ (A344009):

| | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 4 | 6 | 10 | 12 | 18 | 22 | 30 | 34 | 42 | 48 | 58 | 60 | 78 | 82 | 102 | 108 | 118 | 132 | 150 | 154 | 174 | 192 | 210 | 214 | 240 | 258 | 274 | 282 | 322 | 330 |
| 3 | 5 | 8 | 11 | 16 | 20 | 24 | 32 | 36 | 46 | 54 | 59 | 72 | 80 | 90 | 106 | 114 | 120 | 142 | 152 | 168 | 180 | 198 | 212 | 228 | 252 | 270 | 276 | 318 | 324 | 334 | 370 |
| 7 | 9 | 14 | 17 | 23 | 28 | 35 | 40 | 52 | 56 | 70 | 76 | 84 | 94 | 112 | 116 | 138 | 144 | 162 | 172 | 196 | 202 | 222 | 234 | 262 | 272 | 298 | 320 | 332 | 342 | 382 | 396 |
| 13 | 15 | 21 | 26 | 33 | 38 | 47 | 53 | 66 | 71 | 83 | 92 | 107 | 113 | 130 | 140 | 156 | 166 | 190 | 200 | 216 | 232 | 256 | 264 | 288 | 312 | 328 | 336 | 378 | 390 | 416 | 432 |
| 19 | 25 | 29 | 37 | 44 | 50 | 57 | 68 | 77 | 88 | 96 | 110 | 119 | 136 | 148 | 160 | 178 | 197 | 204 | 226 | 250 | 260 | 275 | 294 | 323 | 335 | 358 | 380 | 408 | 419 | 444 | 478 |
| 27 | 31 | 41 | 45 | 55 | 64 | 74 | 81 | 95 | 100 | 117 | 126 | 143 | 155 | 167 | 179 | 203 | 208 | 238 | 251 | 268 | 286 | 316 | 326 | 354 | 359 | 406 | 412 | 436 | 462 | 491 | 503 |
| 39 | 43 | 51 | 62 | 69 | 75 | 93 | 98 | 115 | 124 | 137 | 153 | 164 | 176 | 201 | 206 | 236 | 239 | 263 | 280 | 300 | 317 | 348 | 356 | 392 | 407 | 430 | 452 | 479 | 497 | 533 | 546 |
| 49 | 61 | 65 | 73 | 86 | 97 | 104 | 122 | 131 | 146 | 158 | 173 | 191 | 205 | 215 | 237 | 257 | 278 | 292 | 310 | 333 | 352 | 366 | 400 | 417 | 443 | 474 | 494 | 522 | 544 | 558 | 599 |
| 63 | 67 | 85 | 89 | 101 | 111 | 128 | 134 | 157 | 161 | 186 | 194 | 213 | 233 | 246 | 261 | 287 | 296 | 329 | 346 | 357 | 384 | 413 | 426 | 466 | 476 | 514 | 526 | 556 | 592 | 614 | 626 |
| 79 | 87 | 99 | 105 | 125 | 129 | 149 | 159 | 177 | 188 | 209 | 220 | 244 | 248 | 284 | 293 | 321 | 340 | 353 | 376 | 410 | 415 | 448 | 472 | 501 | 516 | 551 | 574 | 606 | 621 | 659 | 690 |
| 91 | 103 | 121 | 127 | 141 | 151 | 170 | 184 | 195 | 218 | 235 | 245 | 266 | 290 | 299 | 325 | 347 | 371 | 388 | 411 | 431 | 456 | 486 | 512 | 527 | 564 | 596 | 608 | 646 | 682 | 700 | 737 |
| 109 | 123 | 135 | 145 | 165 | 175 | 189 | 199 | 227 | 242 | 265 | 273 | 297 | 311 | 344 | 350 | 377 | 404 | 428 | 437 | 477 | 495 | 520 | 535 | 576 | 597 | 636 | 653 | 692 | 710 | 760 | 770 |
| 133 | 139 | 163 | 171 | 185 | 193 | 224 | 230 | 254 | 269 | 295 | 306 | 338 | 345 | 374 | 383 | 424 | 434 | 473 | 488 | 515 | 530 | 572 | 587 | 625 | 647 | 684 | 704 | 744 | 764 | 796 | 827 |
| 147 | 169 | 182 | 187 | 217 | 225 | 243 | 267 | 281 | 304 | 314 | 341 | 368 | 381 | 405 | 429 | 446 | 484 | 508 | 518 | 560 | 575 | 604 | 640 | 657 | 695 | 715 | 748 | 784 | 810 | 866 | 887 |
| 181 | 183 | 211 | 221 | 231 | 249 | 277 | 285 | 308 | 327 | 364 | 369 | 395 | 422 | 441 | 455 | 489 | 513 | 557 | 568 | 593 | 615 | 655 | 671 | 705 | 743 | 775 | 791 | 835 | 877 | 893 | 911 |
| 207 | 219 | 229 | 247 | 271 | 283 | 305 | 315 | 355 | 365 | 386 | 409 | 435 | 453 | 485 | 493 | 548 | 563 | 580 | 605 | 650 | 666 | 698 | 726 | 771 | 788 | 830 | 869 | 890 | 908 | 976 | 986 |
| 223 | 241 | 255 | 279 | 302 | 309 | 337 | 362 | 375 | 398 | 425 | 449 | 475 | 487 | 517 | 554 | 573 | 595 | 635 | 664 | 688 | 711 | 746 | 785 | 800 | 850 | 884 | 897 | 957 | 977 | 1004 | 1045 |
| 253 | 259 | 291 | 303 | 319 | 343 | 367 | 385 | 421 | 433 | 460 | 482 | 506 | 524 | 566 | 578 | 633 | 638 | 680 | 693 | 741 | 753 | 797 | 812 | 881 | 885 | 923 | 966 | 998 | 1019 | 1070 | 1102 |
| 289 | 301 | 313 | 339 | 363 | 379 | 401 | 427 | 458 | 467 | 505 | 521 | 561 | 577 | 602 | 637 | 668 | 683 | 731 | 749 | 794 | 804 | 851 | 883 | 917 | 935 | 981 | 1016 | 1064 | 1100 | 1133 | 1165 |
| 307 | 331 | 351 | 373 | 393 | 403 | 445 | 464 | 483 | 519 | 542 | 565 | 584 | 629 | 644 | 677 | 701 | 745 | 765 | 803 | 831 | 854 | 895 | 928 | 971 | 1001 | 1031 | 1065 | 1127 | 1139 | 1180 | 1235 |
| 349 | 361 | 389 | 397 | 439 | 457 | 470 | 499 | 531 | 549 | 581 | 590 | 641 | 665 | 686 | 724 | 758 | 782 | 820 | 846 | 891 | 914 | 950 | 974 | 1026 | 1034 | 1115 | 1130 | 1169 | 1209 | 1247 | 1286 |
| 387 | 391 | 423 | 451 | 465 | 471 | 525 | 545 | 579 | 585 | 631 | 662 | 681 | 697 | 747 | 769 | 815 | 844 | 861 | 909 | 940 | 964 | 1017 | 1028 | 1104 | 1118 | 1146 | 1187 | 1244 | 1274 | 1312 | 1346 |
| 399 | 447 | 461 | 469 | 509 | 541 | 555 | 583 | 609 | 651 | 674 | 689 | 734 | 757 | 789 | 824 | 848 | 904 | 921 | 955 | 997 | 1024 | 1061 | 1108 | 1137 | 1178 | 1226 | 1268 | 1295 | 1313 | 1375 | 1425 |
| 459 | 463 | 507 | 523 | 547 | 569 | 603 | 619 | 663 | 675 | 728 | 735 | 787 | 808 | 845 | 849 | 915 | 933 | 989 | 1013 | 1049 | 1084 | 1135 | 1144 | 1205 | 1233 | 1293 | 1300 | 1356 | 1409 | 1446 | 1460 |
| 481 | 511 | 543 | 559 | 591 | 607 | 661 | 669 | 703 | 729 | 761 | 801 | 825 | 847 | 913 | 926 | 968 | 999 | 1035 | 1063 | 1124 | 1142 | 1190 | 1215 | 1281 | 1298 | 1328 | 1401 | 1444 | 1457 | 1535 | 1564 |
| 529 | 553 | 571 | 601 | 639 | 667 | 685 | 722 | 739 | 795 | 818 | 829 | 871 | 925 | 944 | 969 | 1015 | 1051 | 1097 | 1125 | 1154 | 1207 | 1250 | 1289 | 1307 | 1382 | 1423 | 1454 | 1491 | 1544 | 1586 | 1611 |
| 567 | 589 | 627 | 645 | 679 | 699 | 733 | 759 | 806 | 821 | 855 | 902 | 938 | 962 | 1010 | 1025 | 1082 | 1113 | 1148 | 1155 | 1238 | 1275 | 1304 | 1324 | 1405 | 1442 | 1472 | 1517 | 1568 | 1598 | 1665 | 1690 |
| 613 | 643 | 673 | 691 | 725 | 751 | 805 | 813 | 853 | 901 | 937 | 951 | 991 | 1022 | 1071 | 1088 | 1145 | 1149 | 1225 | 1269 | 1297 | 1315 | 1389 | 1411 | 1464 | 1503 | 1545 | 1589 | 1641 | 1686 | 1746 | 1751 |
| 649 | 687 | 721 | 727 | 793 | 809 | 823 | 865 | 905 | 941 | 965 | 1005 | 1039 | 1085 | 1121 | 1147 | 1184 | 1245 | 1285 | 1301 | 1360 | 1395 | 1448 | 1476 | 1528 | 1574 | 1601 | 1658 | 1717 | 1748 | 1785 | 1851 |
| 709 | 723 | 781 | 799 | 817 | 842 | 903 | 919 | 961 | 985 | 1029 | 1059 | 1103 | 1131 | 1167 | 1202 | 1273 | 1291 | 1355 | 1370 | 1431 | 1463 | 1521 | 1538 | 1597 | 1629 | 1701 | 1727 | 1773 | 1839 | 1874 | 1906 |
| 763 | 783 | 811 | 841 | 879 | 907 | 945 | 967 | 1023 | 1047 | 1101 | 1123 | 1159 | 1185 | 1255 | 1279 | 1353 | 1358 | 1399 | 1461 | 1501 | 1526 | 1585 | 1625 | 1691 | 1724 | 1761 | 1831 | 1868 | 1886 | 1977 | 1994 |
| 807 | 819 | 859 | 889 | 939 | 963 | 987 | 1033 | 1083 | 1106 | 1141 | 1181 | 1237 | 1264 | 1299 | 1357 | 1371 | 1445 | 1468 | 1514 | 1562 | 1622 | 1635 | 1694 | 1733 | 1777 | 1845 | 1885 | 1949 | 1988 | 2026 | 2093 |

The sieve of Flavius Josephus (N1048, M2636, A960)

| | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| **1** | ~~2~~ | 3 | ~~4~~ | 5 | ~~6~~ | 7 | ~~8~~ | 9 | ~~10~~ | 11 | ~~12~~ | 13 | ~~14~~ | 15 | ~~16~~ | 17 | ~~18~~ | 19 | ~~20~~ | 21 | ~~22~~ | 23 | ~~24~~ | 25 | ~~26~~ | 27 | ~~28~~ | 29 | ~~30~~ | 31 | ~~32~~ | 33 | ~~34~~ | 35 | ~~36~~ | 37 | ~~38~~ | 39 | ~~40~~ |
| **1** | | **3** | | ~~5~~ | | 7 | | 9 | | ~~11~~ | | 13 | | 15 | | ~~17~~ | | 19 | | 21 | | ~~23~~ | | 25 | | 27 | | ~~29~~ | | 31 | | 33 | | ~~35~~ | | 37 | | 39 | |
| **1** | | **3** | | | | **7** | | ~~9~~ | | | | 13 | | 15 | | | | 19 | | ~~21~~ | | | | 25 | | 27 | | | | 31 | | ~~33~~ | | | | 37 | | 39 | |
| **1** | | **3** | | | | **7** | | | | | | **13** | | ~~15~~ | | | | 19 | | | | | | 25 | | 27 | | | | 31 | | | | | | ~~37~~ | | 39 | |
| **1** | | **3** | | | | **7** | | | | | | **13** | | | | | | **19** | | | | | | ~~25~~ | | 27 | | | | 31 | | | | | | | | 39 | |
| **1** | | **3** | | | | **7** | | | | | | **13** | | | | | | **19** | | | | | | | | **27** | | | | ~~31~~ | | | | | | | | 39 | |
| **1** | | **3** | | | | **7** | | | | | | **13** | | | | | | **19** | | | | | | | | **27** | | | | | | | | | | | | **39** | |

(The first column of $ч^{(\infty)}$ is easily seen to be the result of *this* sieve — whose $n$th term is $(\pi/4)n^2 + O(n^{4/3})$, according to Andersson, 1998.)

Based on the known asymptotics of the top row and leftmost column, and on an examination of the near-right-triangular region of $ч^{(\infty)}$ that contains the numbers $\{1, 2, \ldots, n\}$ for various $n$, Nikolai Beluhov conjectures that $ч^{(\infty)}_{i,j} \approx (\pi i + 2j)^2/(4\pi)$. Indeed, his formula does give a good approximation when $i$ and $j$ are small enough to allow checking by computer.

Everything I know about this subject can be found at

`http://cs.stanford.edu/~knuth/fasc14a.ps.gz`

which is a very preliminary draft of Section 7.5.1 of *The Art of Computer Programming*, “Bipartite matching.”

I ran across Tchoukaillon while trying unsuccessfully to find a bipartite graph for which the Hopcroft–Karp algorithm for optimum matching runs as slowly as its theoretical worst case. One of the main open problems in the theory of bipartite matching is to find lower bounds for the running time of that algorithm; nobody has yet found an infinite family of bipartite graphs for which the algorithm doesn’t run in linear time! The partial results in exercise 14 of my current draft, based on modified Tchoukaillon arrays, might lead to a nonlinear lower bound.

On the other hand, I also kind of wish that the worst case of that algorithm is actually linear.