Amazing grace

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Suppose we assign distinct labels to the vertices of a graph with m edges, where each label is one of the numbers $\{0, 1, 2, \ldots, m\}$. Every edge is then implicitly labeled by the difference between the labels of its endpoints; its label will consequently be between 1 and m. The labeling is called "graceful" if the edge labels, like the vertex labels, are also distinct — or equivalently, if there's exactly one edge with every possible label. The graph is called "graceful" if it has at least one graceful labeling.

These concepts were introduced in 1965, under different names, by Alexander Rosa in his unpublished doctoral thesis [1]. But I first learned of them after Sol Golomb had discovered them independently, and had graciously bestowed on them a terminology that quickly became standard [2]. Sol wrote to Martin Gardner about his findings, and Martin popularized them early in 1972 [3]. (By that time a few bits of Rosa's work had gradually become known to Sol and others on this side of the Iron Curtain.)

Over the years I've been hearing more and more about this topic, especially with respect to the famous/infamous "graceful tree conjecture," namely that the graph of any tree is graceful. This conjecture, still unsolved, is somewhat maddening because just about every other question involving trees has turned out to be fairly easy to answer.

A few months ago I decided to feature graceful graphs as one of the main example applications of techniques for "constraint satisfaction," which will be the subject of Section 7.2.2.3 of *The Art of Computer Programming* [4]. Therefore I began to study Joe Gallian's masterful dynamic survey of the subject [5] in some detail. And I soon learned that Alexander Rosa had done much more than introduce the topic: He had already discovered a significant number of the most important theorems, before publishing anything!

Naturally I wanted to know how he had originally described those early discoveries, in his own words, because graph theory was still pretty much in its infancy during the 1960s. So I wrote to him, asking if there was any way that I could see a copy of [1].

He replied that "it was not published, I typed it myself on a mechanical typewriter in 4 copies of which I have one. I presume another one is in the Mathematical Institute of the Slovak Academy of Sciences in Bratislava." His own copy was, unfortunately, in his university office, which was locked down because of the current pandemic.

I wrote back immediately and said, "When you do get to your office, I hope you'll be able not only to find the thesis, but to make steps to have it scanned and put online. That thesis has inspired a huge amount of work by hundreds of mathematicians all around the world, and such researches show no sign whatsoever of decreasing." And happily, after further nudging, I was delighted to receive scans a couple weeks later from Rosa's colleague Frantisek Franek, from which I could print my own copy. Hurray!

Of course his thesis was written in Slovak, a language that I'd never seen before. But I was glad to see that it was extremely well typed, and beautifully illustrated, and that the language looked rather similar to Russian (which I had studied in college). It turned out, in fact, that 'Google translate' does an excellent job of rendering mathematical Slovak into mathematical English. Thus it was possible to understand virtually everything that he had written, without great difficulty — although I had no idea how to pronounce any of the words properly. I almost came to believe that the Slovak language had been intentionally designed, centuries earlier, to be optimal for mathematical exposition.

The more I was able to decipher, the more impressed and surprised I became, because his thesis included many things that were omitted from the later publications that had supposedly summarized its results. (See [6].) By this time I thought I knew a thing or two about graceful graphs; but he had already discovered considerably more. Thus, on several pages I found myself making a marginal annotation: "Amazing!"

The main purpose of the present note is to describe one of those hitherto unpublished constructions. The topic is beyond the scope of *The Art of Computer Programming*, so I cannot include these observations in that book. Yet I can't resist describing them informally here, because I believe there's a good chance that other researchers will be able to take his appealing ideas and carry them further.

I shall also describe a related construction, which occurred to me while writing my draft, because it is evidently capable of many generalizations that I won't have time to explore personally. Perhaps, dear reader, you are just the right person to take the next steps.

A stronger condition: α-gracefulness. Rosa's thesis [1] and its partial summary [6] introduced several different flavors of gracefulness, of which the "strongest" was applicable only to bipartite graphs. He called it an 'α-labeling' in [6], where his original 'β-labeling' was equivalent to what is now called 'graceful'.*

My favorite way to define α-labeling [4] is somewhat different from Rosa's own version. Given a bipartite graph, let the vertices of one part be given labels 0, 1, …, as before, but let the vertices of the other part be given "barred" labels $\overline{0}$, $\overline{1}$, … . This is an α-labeling if (i) there's exactly one edge $u \mathrel{—} \overline{v}$ such that $u+v=j$, for $0 \le j < m$; and (ii) the largest labels $u_{\max}$ and $\overline{v_{\max}}$ satisfy $u_{\max}+v_{\max}=m-1$.

For example, here's an α-labeling of the path P_8:

$$0 \text{ — } \overline{0} \text{ — } 1 \text{ — } \overline{1} \text{ — } 2 \text{ — } \overline{2} \text{ — } 3 \text{ — } \overline{3}.$$

We clearly have $0+0=0$, $0+1=1$, $1+1=2$, …; and $u_{\max}+v_{\max}=3+3=6=m-1$, because there are $m=7$ edges. Any α-labeling is also a graceful labeling, when we treat each barred label $\overline{v}$ as a representation of $m-v$, since the edge labels are $m=(m-0)-0$; $m-1=(m-1)-0$; …; $1=(m-v_{\max})-u_{\max}$.

Rosa actually gave an elegant α-labeling of the general rectangular grid graph $P_s \,\square\, P_t$, as part A of Lemma 4.3 in his thesis [1, pages 76–77]. It looks like this in my notation, when $s=5$ and $t=8$:

$$\begin{array}{ccccccccccccccc}
0 & — & \overline{0} & — & 1 & — & \overline{1} & — & 2 & — & \overline{2} & — & 3 & — & \overline{3} \\
| & & | & & | & & | & & | & & | & & | & & | \\
\overline{7} & — & 8 & — & \overline{8} & — & 9 & — & \overline{9} & — & 10 & — & \overline{10} & — & 11 \\
| & & | & & | & & | & & | & & | & & | & & | \\
15 & — & \overline{15} & — & 16 & — & \overline{16} & — & 17 & — & \overline{17} & — & 18 & — & \overline{18} \\
| & & | & & | & & | & & | & & | & & | & & | \\
\overline{22} & — & 23 & — & \overline{23} & — & 24 & — & \overline{24} & — & 25 & — & \overline{25} & — & 26 \\
| & & | & & | & & | & & | & & | & & | & & | \\
30 & — & \overline{30} & — & 31 & — & \overline{31} & — & 32 & — & \overline{32} & — & 33 & — & \overline{33}
\end{array} \tag{1}$$

The 67 edges occur nicely in strict order, left to right and top to bottom: $0+0$, $0+1$, …, $3+3$; $0+7$, $0+8$, …, $3+11$; $7+8$, $8+8$, …, $10+11$; $7+15$, …; …, $33+33$. (Gallian's survey [5] is currently unaware of any such construction before 1981. But Rosa's Lemma 4.3 was actually much more general than this.)

Rosa also considered a similar yet quite different family of graphs, in which we take n paths of length r and join them at one end. I shall call this an "(n,r)-star," because I don't know of any existing terminology for such a graph. (In these terms the traditional star graph, $K_{1,n}$, becomes just a special case, the "$(n,1)$-star.") Here is his remarkable construction for the $(5,7)$-star:

$$\begin{array}{ccccccccccccccc}
0 & — & \overline{0} & — & 1 & — & \overline{1} & — & 2 & — & \overline{2} & — & 3 & — & \overline{3} \\
0 & — & 7 & — & \overline{6} & — & 6 & — & \overline{5} & — & 5 & — & \overline{4} & — & 4 \\
0 & — & \overline{7} & — & 8 & — & \overline{8} & — & 9 & — & \overline{9} & — & 10 & — & \overline{10} \\
0 & — & 14 & — & \overline{13} & — & 13 & — & \overline{12} & — & 12 & — & \overline{11} & — & 11 \\
0 & — & \overline{14} & — & 15 & — & \overline{15} & — & 16 & — & \overline{16} & — & 17 & — & \overline{17}
\end{array} \tag{2}$$

[See part I of Lemma 3.3 in [1].] This labeling has edges like '$0 \mathrel{—} 7$', so it's *not* α-graceful. But we shouldn't expect it to be, because Rosa proved that the simplest nontrivial case, the $(3,2)$-star, does not *have* an α-graceful labeling. In fact, he showed that it's actually the *smallest* bipartite graceful graph that is not also α-graceful.

But his labeling of the (n,r)-star is always *graceful*, if we interpret $\overline{v}$ as $m-v$, where $m=nr$ is the number of edges. Indeed, row 1 of his construction covers edges m, $m-1$, …, $m-r+1$ (left to right); then row 2 covers edges $m-r-1$, $m-r-2$, …, $m-2r+1$, and r (right to left); then row 3 covers edges $m-r$ and $m-2r-1$, $m-2r-2$, …, $m-3r+1$ (left to right); and so on. In general, row $2k+1$ covers edges

* In [1], the roles of α and β were actually reversed; Rosa evidently decided in 1966 that the opposite convention would be better. Fortunately that hasn't led to confusion, because nobody actually read [1] between 1966 and 2020! Meanwhile many interesting papers have developed the theory of α-graceful graphs, as discussed in [5], and the terminology of [6] has become standard.

$m-kr$ and $m-2kr-1$ through $m-(2k+1)r+1$ (left to right), while row $2k$ covers edges $m-(2k-1)r-1$ through $m-2kr+1$ and kr (right to left).

So he has clearly covered all edges that are not divisible by r. The multiples of r are, however, trickier: We get m in row 1, $m-r$ in row 3, $m-2r$ in row 5, etc.; we get r in row 2, $2r$ in row 4, $3r$ in row 6, etc. And, aha, yes — all are covered exactly once, when we reach row n. Amazing!

Rosa never got around to publishing the fact that every (n,r)-star is graceful. And despite the tremendous growth in graceful-graph studies, surveyed so comprehensively in [5], it seems that nobody else has ever rediscovered this result, except with serious restrictions on the value of r mod 4.

But Rosa's Lemma 3.3 actually proved much more! In part II, he showed that, instead of joining n isomorphic paths at point 0, we can actually join n isomorphic "caterpillars" at that point; the result will still be graceful. (We'll say a lot more about caterpillars below.) And of course that's even *more* amazing.

In fact, a closer look shows that considerably more is also true — because there really is very little interaction between different rows, in the way we've proved the validity of his construction.

For example, notice that the edge '$1 — \overline{1}$' in the first row of (2) could be replaced by '$2 — \overline{0}$', without messing up gracefulness of the result. (That means the graph of the first row would be a certain tree, instead of a path.) Independent of this, the edge '$\overline{6} — 6$' in row 2 could also be replaced by '$\overline{5} — 7$'.

Similarly, back in row 1, '$\overline{1} — 2$' could be replaced by either '$\overline{2} — 1$' or '$\overline{3} — 0$'; '$2 — \overline{2}$' could be replaced by either '$1 — \overline{3}$' or '$3 — \overline{1}$'; and '$\overline{2} — 3$' could be replaced by '$\overline{3} — 2$'. That gives $2\cdot 3\cdot 3\cdot 2 = 36$ different ways to label the vertices in row 1 while preserving gracefulness. Independently, the same reasoning leads to 36 different ways to label the vertices in row 2, etc., making 36^n graceful possibilities in all.

Let's say that an α-labeling is *pendant* if vertex 0 appears in only one edge. And let's say that two α-graceful graphs are *compatible* if they have α-labelings with the same values of $u_{\max}$ and $v_{\max}$. (Notice that compatible graphs always have the same number of edges, r, because $u_{\max}+v_{\max}=r-1$.) In these terms, Rosa's amazing construction has a far-reaching generalization:

Theorem 1. *The graph obtained by joining together any n mutually compatible, pendant α-graceful graphs at vertex 0 is graceful.*

Proof. Let the given graphs have r edges each, and let $m=nr$. In the kth α-graceful labeling, for $1\le k\le n$, change the labels of all vertices *except* vertex 0, as follows: If $k=2j+1$, change u to $u+jr$ and $\overline{v}$ to $\overline{v+jr}$; but if $k=2j$, change u to $\overline{jr-u}$ and $\overline{v}$ to $jr-v$. If we now interpret $\overline{v}$ as $m-v$, the r edges in graph k run through the same set of values as the edges in row k of Rosa's construction for (n,r)-stars. ▮

I like to think that the extra generality revealed by this proof is a consequence of the new notation for α-labeling that I've been using. Otherwise the essential structure might well have remained hidden.

When applying this theorem, it's sometimes helpful to use a nice fact that Rosa noted in Lemma 3.4 of [1]: Every α-labeling has an *inverse*, which reverses the order of the edge labels. (D. A. Sheppard subsequently learned this fact from Rosa, and called it the "edge complement" [7]; but it remains little known.) The proof is simple: We simply change each label u to $u_{\max}-u$ and each label $\overline{v}$ to $\overline{v_{\max}-v}$. Notice that vertex 0 might be pendant in the inverse but not in the original labeling.

An attentive reader might have noted that Theorem 1 has not actually exploited all of the flexibility that exists in these constructions. For example, there's no need for vertex 0 to be pendant in the very first α-graceful graph. Moreover, suppose we alter (2) by changing the edges of row 1 to

$$0 — \overline{0} — 1 — \overline{1} — 3 — \overline{3} \qquad \text{and} \qquad 1 — \overline{2} — 3.$$

This graph now contains a 4-cycle, and it has no vertex labeled 2. Therefore we could, for instance, delete edge '$\overline{4} — 4$' from row 2 and replace it with the new edge '$2 — \overline{6}$'. And then we could change '$3 — \overline{3}$' to '$\overline{2} — 4$', etc. All of these changes preserve gracefulness as the graphs evolve.

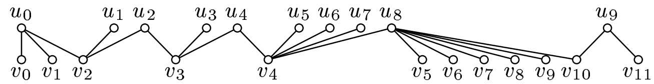
Thus I look forward to being quite happily amazed by future generalizations of Theorem 1.

Caterpillar nets. Let's turn now to a special class of graphs whose α-labelings are particularly simple. As far as I know, Rosa was the first to single these graphs out for special attention, although without naming them [1, Lemma 3.1; 6, Theorem 2]; the same graphs came up also a few years later in several quite different contexts [8].

A "caterpillar" is a graph with at least two vertices that becomes a path (or empty) when you remove all of its vertices of degree 1. More precisely, an (s,t)-*caterpillar* is a bipartite graph with vertices $\{u_0,\ldots,u_s;v_0,\ldots,v_t\}$ and edges defined by a binary vector $e=e_1\ldots e_{s+t}$ that has s 0s and t 1s:

$$u_{s_i} \text{---} v_{t_i} \quad \text{for } 0\le i\le s+t, \quad \text{where } s_i=\bar e_1+\cdots+\bar e_i \text{ and } t_i=e_1+\cdots+e_i. \tag{3}$$

(In a binary vector, $\bar x$ stands for $1-x$.) For example, here's the $(9,11)$-caterpillar whose edge vector e is 11001001000011111101:



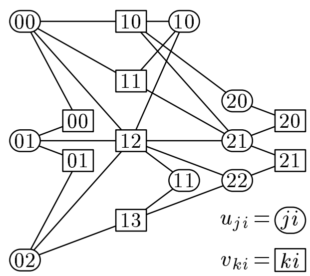
(4)

The edge vector also encodes the α-labeling, which in this case is

$$u_0 \text{---} v_0,\ u_0 \text{---} v_1,\ u_0 \text{---} v_2,\ u_1 \text{---} v_2,\ u_2 \text{---} v_2,\ u_2 \text{---} v_3,\ u_3 \text{---} v_3,\ u_4 \text{---} v_3,\ \ldots,\ u_9 \text{---} v_{11}.$$

Given an (s,t)-caterpillar, a "caterpillar net" is a graph obtained when we replace its vertices u_j and v_k by disjoint *sets* of vertices $U_j=\{u_{j0},\ldots,u_{jp_j}\}$ and $V_k=\{v_{k0},\ldots,v_{kq_k}\}$, for $0\le j\le s$ and $0\le k\le t$. The edges are (p_{s_i},q_{t_i})-caterpillars between U_{s_i} and V_{t_i}, for $0\le i\le s+t$.

For example, here's a caterpillar net with $e=1001$, $p_0=p_2=2$, $p_1=q_0=q_2=1$, and $q_1=3$:



(5)

Oval boxes indicate elements of U_0, U_1, and U_2; rectangular boxes indicate elements of V_0, V_1, and V_2.

In general, a caterpillar net has $m=m_{s+t+1}$ edges, where $m_0=0$ and $m_{i+1}=m_i+p_{s_i}+q_{t_i}+1$. So in this particular case we have $m_1=p_0+q_0+1=4$ edges between U_0 and V_0; $m_2=m_1+p_0+q_1+1=10$ edges between U_0 and $V_0\cup V_1$; $m_2=m_2+p_1+q_1+1=15$ edges between $U_0\cup U_1$ and $V_0\cup V_1$; etc.

Theorem 2. *Every caterpillar net is α-graceful.*

Proof. Assign the label a_j+i to each element u_{ji} of U_j, and the label $\overline{b_k+i}$ to each element v_{ki} of V_k, where

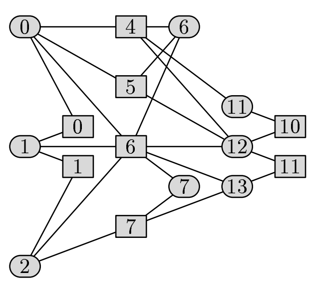
$$a_0=b_0=0 \qquad \text{and} \qquad a_{j+1}=a_j+p_j+q_{j'}+1, \quad b_{k+1}=b_k+p_{k''}+q_k+1; \tag{6}$$

here $j'=\max\{i\mid u_j \text{---} v_i\}$ and $k''=\max\{i\mid u_i \text{---} v_k\}$. The labels are distinct because $a_{j+1}>a_j+p_j$ and $b_{k+1}>b_k+q_k$.

These definitions ensure that $a_{s_i}+b_{t_i}=m_i$. Hence the edges $u \text{---} \overline{v}$ of the caterpillar between U_{s_i} and V_{t_i} account for the sums $u+v$ that successively equal m_i, m_i+1, …, $m_{i+1}-1$ in part (i) of the definition of α-gracefulness.

When $i=s+t$ we have $s_i=s$ and $t_i=t$. The final edge goes from $u_{\max}=a_s+p_s$ to $\overline{v_{\max}}=\overline{b_t+q_t}$; and $a_s+p_s+b_t+q_t=m-1$. This establishes part (ii) of the definition. ▮

In the example, we have $(a_0,a_1,a_2)=(0,6,11)$ and $(b_0,b_1,b_2)=(0,4,10)$:



(7)

(Here label v inside a rectangular box stands for $\overline{v}$.)

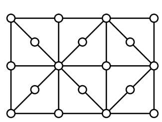
The simplest applications of Theorem 2 occur when $s = 0$, for in such cases the "base" caterpillar that underlies the caterpillar net is simply the star graph $K_{1,t+1}$. If we also set $q_0 = \cdots = q_t = 0$, each of the caterpillars between U_0 and V_i is also a star graph, $K_{p_0+1,1}$. The resulting caterpillar net is therefore the complete bipartite graph $K_{p_0+1,t+1}$! And Theorem 2 gives it an α-labeling equivalent to another construction found in Rosa's early work [1, Lemma 4.1; 6, Theorem 6]. Of course many further interesting and visually appealing cases come to light when we use larger values of $q_0, q_1, \ldots,$ even when $s = 0$.

Another straightforward application of Theorem 2 arises when the base caterpillar is simply a path, $U_0 - \overline{V_0} - U_1 - \overline{V_1} - U_2 - \cdots$, and when the caterpillars that connect U_i to V_i to U_{i+1} are themselves paths, such as the "staircase paths" in the following α-labeling of $P_5 \mathbin{\square} P_8$:

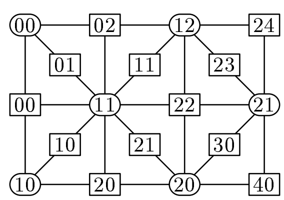
$$\begin{array}{ccccccccccccccc}
0 & - & \overline{1} & - & 4 & - & \overline{7} & - & 12 & - & \overline{16} & - & 21 & - & \overline{25} \\
| & & | & & | & & | & & | & & | & & | & & | \\
\overline{0} & - & 3 & - & \overline{6} & - & 11 & - & \overline{15} & - & 20 & - & \overline{24} & - & 29 \\
| & & | & & | & & | & & | & & | & & | & & | \\
2 & - & \overline{5} & - & 10 & - & \overline{14} & - & 19 & - & \overline{23} & - & 28 & - & \overline{31} \\
| & & | & & | & & | & & | & & | & & | & & | \\
\overline{4} & - & 9 & - & \overline{13} & - & 18 & - & \overline{22} & - & 27 & - & \overline{30} & - & 33 \\
| & & | & & | & & | & & | & & | & & | & & | \\
8 & - & \overline{12} & - & 17 & - & \overline{21} & - & 26 & - & \overline{29} & - & 32 & - & \overline{33}
\end{array} \qquad (8)$$

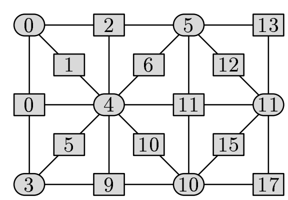
This labeling, due to B. D. Acharya and M. K. Gill [9], is actually what led me to think of Theorem 2 in the first place. (In fact, I thought for awhile that it was the "only" simple way to label a rectangular grid graph — until I read [1] and learned of Rosa's even simpler construction, which yields (1).)

Theorem 2 allows us to go much further, for example to "decorate" the grid by inserting internal edges, as in the following example:



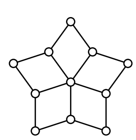
labeled by



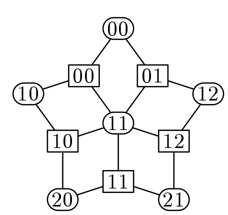


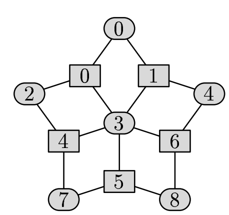
(9)

We can also make a flower with five petals:



labeled by





(10)

and extend it easily to six, seven, or any larger number of petals.

I wrote a simple program that tries to see if a given graph is a caterpillar net, and of course I fed in graph (5) as a test case. The computer astonished me by reporting that I could also have started with the completely different base caterpiller

$$u_0 - v_0, \quad u_1 - v_0, \quad u_1 - v_1, \quad u_1 - v_2, \quad u_2 - v_2, \quad u_2 - v_3.$$

Replace u_0 by $\{(20)\}$; replace v_0 by $\{[10],[20]\}$; replace u_1 by $\{(00),(10),(21)\}$; replace v_1 by $\{[11]\}$; replace v_2 by $\{[00],[12],[21]\}$; replace u_2 by $\{(01),(11),(02),(22)\}$; and replace v_3 by $\{[01],[13]\}$. Add the edges we had before. They now form caterpillars between these new U's and V's! It must be simply a weird coincidence.

The number of possibilities that Theorem 2 can deal with is quite mind-boggling. There must be many genera and species of "wowie" caterpillar nets.

Yet, as in Theorem 1, Theorem 2 doesn't fully exploit the flexibility of the underlying proof. For example, it's necessary to prove that $a_{j+1} > a_j + p_j$ and that $b_{k+1} > b_k + q_k$; but in fact those inequalities aren't always very "tight." We can often succeed without using a full caterpillar to connect U_{s_i} with V_{t_i}, if we connect together only an "interval" of those vertices instead.

Consider the construction of $K_{4,4}$ above, but remove some of the edges so that, instead of full caterpillars between U_0 and V_j, we use only certain subintervals. We might, for instance, retain only 9 of the original 16 edges:

$$v_{00} \,—\, u_{00}, u_{01}, u_{02}; \qquad v_{10} \,—\, u_{01}, u_{02}; \qquad v_{20} \,—\, u_{01}, u_{02}, u_{03}; \qquad v_{30} \,—\, u_{03}.$$

This graph is α-graceful, because we can give the labels 0, 1, 2, 3 to u_{00}, u_{01}, u_{02}, u_{03} while labeling v_{00}, v_{10}, v_{20}, v_{30} with $\overline{0}$, $\overline{2}$, $\overline{4}$, $\overline{5}$.

Examples are also easy to construct where we chop lots of edges out of a rectangular grid.

But I don't see a clean way to formulate the extra generality that's obtainable with such "subinterval caterpillars." (In addition to proving that $a_{j+1} > a_j + p_j$ and $b_{k+1} > b_k + q_k$, one must also ensure that $u_{\max} + v_{\max} = m - 1$.)

Maybe it's also possible to combine Theorem 1 with Theorem 2. Who knows?

[1] Alexander Rosa, *O Cyklických Rozkladoch Kompletného Grafu*, Kandidátska dizertačná práca (Bratislava: Československá akadémia vied (November 1965), ii + 86 pages. Scans of this document can be viewed online in the following four files, all in directory `http://cs.stanford.edu/~knuth/`: `A.Rosa-thesis-page0-20.pdf`, `A.Rosa-thesis-page21-42.pdf`, `A.Rosa-thesis-page43-63.pdf`, and `A.Rosa-thesis-page64-86.pdf`.

[2] Solomon W. Golomb, "How to number a graph," in *Graph Theory and Computing*, edited by Ronald C. Read (New York: Academic Press, 1972), 23–37.

[3] Martin Gardner, "Mathematical games: The graceful graphs of Solomon Golomb, or how to number a graph parsimoniously," *Scientific American* **226**, 3 (March 1972), 108–112. Reprinted with additions in his book *Wheels, Life and Other Mathematical Amusements* (New York: W. H. Freeman, 1983), Chapter 15.

[4] Donald E. Knuth, preliminary drafts entitled "Constraint satisfaction," posted at `http://cs.stanford.edu/~knuth/fasc7a.ps.gz` and updated frequently as more material is being accumulated.

[5] Joseph A. Gallian, "A dynamic survey of graph labeling," *Electronic Journal of Combinatorics*, #DS6. Version 0 (11 February 1998) had 43 pages and 284 references; he has updated it annually, and Version 23 (17 December 2020) has 553 pages and 2922 references.

[6] A. Rosa, "On certain valuations of the vertices of a graph," in *Theory of Graphs*, the proceedings of an international symposium, Rome, 1966 (Paris: Dunod, 1967), 349–355. [This publication summarized many of the "graceful graph" aspects of [1]; other papers summarized other aspects. For example, he wrote "Poznámka o cyklických Steinerových systémoch trojíc," in *Matematicka-Fyzikálny Časopis Slovenskeij Akadémie Vied* **16** (1966), 53–63. Also "On the decompositions of the complete graph into $(4m + 2)$-gons," in *Matematicka-Fyzikálny Časopis Solvenskeij Akadémie Vied* **16** (1966), 349–352.]

[7] David A. Sheppard, "The factorial representation of balanced labelled graphs," *Discrete Mathematics* **15** (1976), 379–388.

[8] Frank Harary and Allen Schwenk, "Trees with hamiltonian square," *Mathematika* **18** (1971), 138–140.

[9] B. D. Acharya and M. K. Gill, "On the index of gracefulness of a graph and the gracefulness of two-dimensional square lattice graphs," *Indian Journal of Mathematics* **23** (1981), 81–94.