## Ambidextrous Numbers

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Rossi and Thuswaldner [7] recently introduced a fascinating new mathematical object $\mathbb{K} = \mathbb{R} \times \mathbb{Q}_2$, where $\mathbb{R}$ is the field of real numbers and $\mathbb{Q}_2$ is the field of 2-adic numbers. It's a metric space that's closed under the operations of addition, negation, halving, and taking limits. It's somewhat analogous to the complex numbers, because each member of $\mathbb{K}$ has a "real part" and a "2-adic part."

I think $\mathbb{K}$ is potentially very important; therefore it needs a good name. What should we call its elements? My first inclination was to call them "biadic numbers," because $\mathbb{K}$ is bipartite and one of the parts is 2-adic. But Google told me that that term has already been co-opted in coding theory. (When codes are formed from quaternary rings $\mathrm{GR}(4^n)$ instead of from finite fields $\mathrm{GF}(2^n)$, the codewords have a so-called biadic representation.)

After noticing that properties of $\mathbb{K}$ involve both left-hand and right-hand rules (see below), my next idea was to perhaps call its elements "ambiatic numbers." But Google quickly nixed that one too: I soon learned that, these days, ambiatic is an altersex identity.

Fortunately there's at least one nice candidate left. Let's be bold and call those elements *ambinumbers*! That term seems to be not only appropriate, and catchy; it also has apparently never appeared before on the Internet. Furthermore it can easily be translated into other languages.

I'm not sure that $\mathbb{K}$ will turn out to be the best *symbolic* name for the ambinumbers; only time will tell. The letter $K$ might suggest that they form a field, but $\mathbb{K}$ is *not* a field. (For example, $(1,0)$ times $(0,1)$ is $(0,0)$.) If the name "ambinumbers" ever becomes wildly popular, the symbol $\mathbb{A}$ might well be better than $\mathbb{K}$. Meanwhile $\mathbb{K}$ certainly isn't a bad choice.

**1. Binary representation of ambinumbers.** Let $(x,q)$ be an ambinumber, where $x$ is real and $q$ is 2-adic. The fundamental operations of addition, negation, and halving are defined componentwise:

$$(x,q) + (x',q') = (x+x', q+q'); \qquad -(x,q) = (-x,-q); \qquad (x,q)/2 = (x/2, q/2). \tag{1.1}$$

The ring $\mathbb{Z}[\frac{1}{2}]$ of dyadic rational numbers is the smallest subset of $\mathbb{R}$ that contains 0 and also contains $x+1$, $-x$, and $x/2$ whenever it contains $x$. Furthermore $\mathbb{Z}[\frac{1}{2}]$ is also the smallest subset of $\mathbb{Q}_2$ that satisfies those same properties. It's natural to represent an arbitrary element of $\mathbb{Z}[\frac{1}{2}]$ as a finite sum in the binary number system, together with a sign, namely as

$$x \;=\; \bigl(2^l x_l + 2^{l+1} x_{l+1} + \cdots + 2^u x_u\bigr) \;=\; \pm(x_u \ldots x_0.x_{-1} \ldots x_l)_2, \tag{1.2}$$

where $l \le 0 \le u$ and each digit $x_j$ is either 0 or 1. For example,

$$\widehat{\pi} \;=\; (0011.00100100)_2 \tag{1.3}$$

is a dyadic rational approximation to the irrational number

$$\pi \;=\; (0011.0010010000111111011010 \ldots)_2. \tag{1.4}$$

Addition, negation, and halving are well known to be convenient and efficient when binary notation is used.

However, 2-adic numbers are quite different from real numbers. On intuitive grounds, it is sometimes useful to think of a 2-adic number $x$ in terms of its left-right reversal,

$$x^R \;=\; 2^{-l} x_l + 2^{-l-1} x_{l+1} + \cdots + 2^{-u} x_u \;=\; (x_l \ldots x_0.x_1 \ldots x_u)_{1/2}, \tag{1.5}$$

using radix $1/2$ instead of radix 2. The reason is that the idea of "closeness" is quite different in $\mathbb{Q}_2$ than it is in $\mathbb{R}$. For instance, the dyadic rational $(00.10100100)_2$ is a much better 2-adic approximation to $\widehat{\pi}$ than, say, $(11.00101)_2$ is, and this change in viewpoint becomes clearer when we look at the reversals $(1001010.)_{1/2}$, $(101001.1)_{1/2}$, and $\widehat{\pi}^R = (1001001.1)_{1/2}$. (Of course $\pi$ itself has no reversal, in the real number system. But $\pi^R = (11.00100100\ldots)_{1/2}$ is a perfectly respectable 2-adic number, because the series in (1.5) always converges 2-adically as $l \to -\infty$.)

Incidentally, it's interesting to experiment with addition of reversed numbers. When the radix is $1/2$, "carries" propagate to the *right* instead of to the left, so it's easiest to work from left to right.

The representation in (1.2) is obviously not unique, because we can pad it with zeroes at the left and/or at the right. We could make it unique by requiring $x_u = 1$ when $u > 0$, also requiring $x_l = 1$ when $l < 0$; furthermore, if $l = u = x_0 = 0$, the sign should be '+'.

We get the real numbers that *aren't* dyadic rationals by allowing $l$ to become $-\infty$, and taking the limit of the sums in (1.2). The real distance between $x$ and 0 is the absolute value $|x|$, obtained by ignoring the sign.

Switching between left and right, we get the 2-adic numbers that aren't dyadic rationals by allowing $u$ to become $+\infty$, and taking an appropriate limit. The 2-adic distance between $x$ and 0 is defined to be

$$|x|_2 = 2^{-l}, \text{when } x = 2^l + 2^{l+1}x_{l+1} + 2^{l+1}x_{l+2} + \cdots; \ |0|_2 = 0. \tag{1.6}$$

Therefore we have $|2x|_2 = |x|_2/2$.

Convergence to the left and convergence to the right are equally important for ambinumbers. So we define the distance between $(x, y)$ and $(0, 0)$ in $\mathbb{K}$ to be

$$\|(x,y)\| = \max\bigl(|x|, |y|_2\bigr). \tag{1.7}$$

One interesting consequence is that $\|(x,x)\|$ is an integer whenever $x \in \mathbb{Z}[\frac{1}{2}]$.

The *negative* of a 2-adic number with $x_l = 1$ is obtained by simply changing $x_j$ to $1 - x_j$ for $l < j < \infty$. For example,

$$-(10110.011)_2 \ = \ (\ldots 111101001.101)_2. \tag{1.8}$$

This definition works because it clearly makes $x + (-x) = 0$. Therefore 2-adic numbers need no sign.

The reversal mapping (1.5) from 2-adics to reals satisfies the triangle inequality

$$(x+y)^R \le x^R + y^R. \tag{1.9}$$

So it could be used as an alternative norm for 2-adic numbers (and it would define the same topology). It's easy to see that

$$(2x)^R = x^R/2 \qquad \text{and} \qquad |x|_2 \le x^R \le 2|x|_2. \tag{1.10}$$

Exercise 4.1–31 of [3] shows that $x^R$ is rational if and only if the 2-adic number $x$ is rational, and explains how to compute it in the rational case. For example, we have

$$\begin{array}{llllll}
(-1)^R = 2, & (-\frac{2}{3})^R = \frac{2}{3}, & (-\frac{1}{3})^R = \frac{4}{3}, & (0)^R = 0, & (\frac{1}{3})^R = \frac{5}{3}, & (\frac{2}{3})^R = \frac{5}{6}, \\
(-\frac{5}{6})^R = \frac{7}{3}, & (-\frac{3}{5})^R = \frac{6}{5}, & (-\frac{1}{4})^R = 4, & (\frac{1}{6})^R = \frac{10}{3}, & (\frac{2}{5})^R = \frac{7}{10}, & (\frac{3}{4})^R = 6, \\
(-\frac{4}{5})^R = \frac{2}{5}, & (-\frac{1}{2})^R = 1, & (-\frac{1}{5})^R = \frac{8}{5}, & (\frac{1}{5})^R = \frac{7}{5}, & (\frac{1}{2})^R = 2, & (\frac{4}{5})^R = \frac{7}{20}, \\
(-\frac{3}{4})^R = 3, & (-\frac{2}{5})^R = \frac{4}{5}, & (-\frac{1}{6})^R = \frac{8}{3}, & (\frac{1}{4})^R = 4, & (\frac{3}{5})^R = \frac{9}{5}, & (\frac{5}{6})^R = \frac{11}{3}.
\end{array}$$

It's well known that the binary representation of a real number is almost unique, if we insist that $l = -\infty$ and that $x_u = 1$ when $u > 0$. Uniqueness fails if and only if $x$ is a nonzero element of $\mathbb{Z}[\frac{1}{2}]$; in that case, $x$ has exactly two representations, one of which ends with infinitely many 0s and the other ends with infinitely many 1s. Consequently we can have $x_1^R = x_2^R$ for two distinct 2-adic numbers $x_1$ and $x_2$ if and only if $x_1^R$ is a nonzero dyadic rational.

It follows from these remarks that every point of the upper half plane, when represented in Cartesian coordinates $(x, y)$ with $y \ge 0$, corresponds to either one or two ambinumbers $(x, q)$, where $y = q^R$. This gives us a way to sort-of visualize ambinumbers, respecting their "closeness" within a factor of 2.

However, the mapping $(x, q) \mapsto (x, q^R)$ is somewhat peculiar and hard to get used to. For example, the image point $\bigl(1/2^n, 1/(2^n + 1)\bigr)$, which is near zero when $n$ is large, corresponds to the ambinumber $\bigl(1/2^n, -2^n/(2^{2n} + 1)\bigr)$. Similarly, $\bigl(1/2^n, 1/(2^n - 1)\bigr)$ corresponds to $\bigl(1/2^n, -2^n/(2^n - 1)\bigr)$.

**2. Negasemiternary representation of ambinumbers.** Rossi and Thuswaldner discovered that ambinumbers can be represented in a much more insightful way if we express them via the unusual radix $-3/2$, using ternary digits(!). Namely, we can write

$$x = (-\tfrac{3}{2})^u x_u + (-\tfrac{3}{2})^{u-1} x_{u-1} + \cdots + (-\tfrac{3}{2})^l x_l = (x_u \ldots x_0 . x_{-1} \ldots x_l)_{-3/2}, \qquad (2.1)$$

where $l \le 0 \le u$ and each digit $x_j$ is 0 or 1 or 2. For example,

$$(120.1)_{-3/2} = 9/4 - 6/2 - 2/3 = -17/12. \qquad (2.2)$$

Let's call $(x_u \ldots x_0 . x_{-1} \ldots x_l)_{-3/2}$ an "NST number," where NST stands for "negasemiternary." Latinwise, 'nega' means −, 'semi' means 1/2, 'tern' means 3, and 'ary' means radix notation.

When $l$ gets smaller and smaller and approaches $-\infty$, NST numbers converge nicely in both the real and 2-adic contexts, because $\left|(-\frac{3}{2})^l\right| = 3^l/2^l$ and $\left|(-\frac{3}{2})^l\right|_2 = 2^l$. The main theorem of [7] is that every ambinumber can be represented as $(x, x)$ for some infinite-precision NST number $x = (x_u \ldots x_0 . x_{-1} \ldots)_{-3/2}$, and that in fact this representation is almost always unique.

(Having three distinct digits sort of makes sense, because each digit contributes $\lg \frac{3}{2}$ bits of information about the real part and one bit of information about the 2-adic part, hence $\lg 3$ bits altogether.)

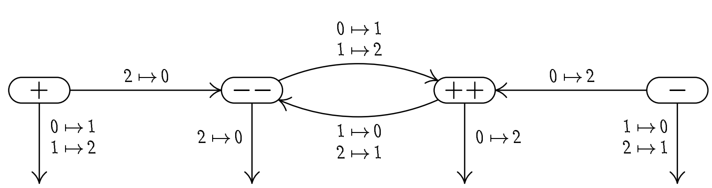
NST numbers are the main characters of this story, so it will be convenient to write them with a special font. *We shall therefore use the back-slanted digits* '0', '1', *and* '2' whenever referring to an explicit NST number. For example, the left-hand side of (2.2) will be written '120.1'. In this way we won't have to bother enclosing the digits inside a '$(\ldots)_{-3/2}$' wrapper. (For similar reasons, I've used typewriter type to distinguish numbers in hexadecimal notation from numbers in decimal notation, and italic digits to signify octal numbers, ever since the first edition of [3] came out in 1969. The font used here for NST numbers is `cmff10`, which has been part of the standard releases of TeX ever since 1982; therefore it's probably already installed on every mathematician's computer.)

We might think at first that NST representations of integers greater than 2 are pretty messy, because powers of $-3/2$ give us fractions whose denominators are hard to cancel out. But the digit 2 comes to our rescue. For example, 20 is $-3$; therefore 21 $= -2$ and 22 $= -1$. Of course we omit the radix point when $l = 0$ in (2.1). A bit of further experimentation reveals that every integer actually has a fairly short representation, and we're soon led to a tapestry of intricate patterns:

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $-27 =$ | 21122000 | $-18 =$ | 21120100 | $-9 =$ | 211200 | $0 =$ | 0 | $9 =$ | 21100 | $18 =$ | 2112200 | $27 =$ | 211201000 |
| $-26 =$ | 21122001 | $-17 =$ | 21120101 | $-8 =$ | 211201 | $1 =$ | 1 | $10 =$ | 21101 | $19 =$ | 2112201 | $28 =$ | 211201001 |
| $-25 =$ | 21122002 | $-16 =$ | 21120102 | $-7 =$ | 211202 | $2 =$ | 2 | $11 =$ | 21102 | $20 =$ | 2112202 | $29 =$ | 211201002 |
| $-24 =$ | 21122210 | $-15 =$ | 211010 | $-6 =$ | 2110 | $3 =$ | 210 | $12 =$ | 2112010 | $21 =$ | 2110110 | $30 =$ | 211201210 |
| $-23 =$ | 21122211 | $-14 =$ | 211011 | $-5 =$ | 2111 | $4 =$ | 211 | $13 =$ | 2112011 | $22 =$ | 2110111 | $31 =$ | 211201211 |
| $-22 =$ | 21122212 | $-13 =$ | 211012 | $-4 =$ | 2112 | $5 =$ | 212 | $14 =$ | 2112012 | $23 =$ | 2110112 | $32 =$ | 211201212 |
| $-21 =$ | 21120120 | $-12 =$ | 211220 | $-3 =$ | 20 | $6 =$ | 21120 | $15 =$ | 2112220 | $24 =$ | 211201020 | $33 =$ | 211222120 |
| $-20 =$ | 21120121 | $-11 =$ | 211221 | $-2 =$ | 21 | $7 =$ | 21121 | $16 =$ | 2112221 | $25 =$ | 211201021 | $34 =$ | 211222121 |
| $-19 =$ | 21120122 | $-10 =$ | 211222 | $-1 =$ | 22 | $8 =$ | 21122 | $17 =$ | 2112222 | $26 =$ | 211201022 | $35 =$ | 211222122 |

One of the first patterns we might notice, for example, is the fact that the units digit, $x_0$, is equal to $x \bmod 3$. Furthermore, the number of digits, $u + 1$, is even if and only if $x < 0$. (Both of these properties hold also in the classical *negaternary* number system, whose radix is $-3$.)

**3. Algorithms.** Given the NST representation of an integer $x$, there's a nice finite-state transducer that produces the representation of $x + 1$, as well as a dual one that produces the representation of $x - 1$:




(3.1)

The algorithm works from right to left, starting with the rightmost digit $x_0$ and then proceeding if necessary to $x_1$, $x_2$, and so on. To add 1, we start at state '+'; to subtract 1, we start at state '−'.

More precisely, if we want to add 1, we simply increase $x_0$ by 1 and stop, unless $x_0 = \mathtt{2}$. In the latter case, we set $x_0 \leftarrow 0$ and move to the left, also going into state '$--$'. In state '$--$', which incidentally happens to be the entry point for a finite-state transducer that subtracts 2 from a given integer $x_u \ldots x_1$, we'll simply set $x_1 \leftarrow 0$ and stop, if $x_1 = \mathtt{2}$; otherwise we increase $x_1$ by 1 and move to the left, also going into state '++'. In state '++', similarly, we simply set $x_2 \leftarrow 2$ and stop, if $x_2 = \mathtt{0}$; otherwise we decrease $x_2$ by 1, move another step left, and go back to state '$--$'. If we ever have to move to the left of $x_u$, we imagine that digit $x_u$ is preceded by infinitely many zeroes. (Equivalently, we could set $x_{u+1} \leftarrow \mathtt{0}$ and $u \leftarrow u + 1$.) Eventually we'll get to a case where the current digit is $x_j = \mathtt{2}$ and $j$ is odd, or $x_j = \mathtt{0}$ and $j$ is even.

These algorithms for adding ±1 or ±2 can be described by simple code in any C-like language:

```
void NSTadvance(int delta) {      // add delta to the NST number x, where |delta| < 3
  register int p, state;
  for (p = 0, state = delta;  state;  p++) {
    x[p] = x[p] + state;
    if (x[p] > 2) x[p] = x[p] − 3, state = +2;
    else if (x[p] < 0) x[p] = x[p] + 3, state = −2;
    else state = 0;                                                          (3.2)
  }
  while (u > 0 && x[u] ≡ 0) u−−;
  while (x[u + 1]) {
    if (++u > precision − 3) exit(OVERFLOW);
  }
}
```

(Here $x$ and $u$ are global variables, and the NST number $x$ is $(x[u] \ldots x[1]x[0])_{-3/2}$. The elements of array $x$ are $x[0]$, $x[1]$, …, $x[precision - 1]$, and we assume that $u \le precision - 3$.)

The table of NST numbers above also exhibits a somewhat more subtle pattern, which can be seen when we add together the digits of $x$ and $-x$ without doing any "carrying." For example, the NST representations of 13 and −13 sum to 2112011 + 211012 = 2323023. We get one or more pairs '23', possibly separated by strings of zeroes! And whenever the pair '23' occurs before carrying, we know that we can replace it by '00', because the radix is −3/2. Thus $x + (-x)$ is clearly zero.

It's not difficult to prove that this pairing phenomenon occurs in general. Hence there's an extremely simple way to negate any given NST number $x = (x_u \ldots x_0)_{-3/2}$:

```
void NSTnegate(void) {      // change the NST number x to −x
  register int p, state;
  for (p = 0;  p ≤ u;  p++) {
    if (x[p] ≡ 0) continue;
    x[p] = 3 − x[p], p++, x[p] = 2 − x[p];
  }                                                    (3.3)
  if (u > 0 && x[u] ≡ 0) u−−;
  else if (x[u + 1]) {
    if (++u > precision − 3) exit(OVERFLOW);
  }
}
```

Speaking of carrying, here's a program that does addition in general. Each of the NST numbers $x$, $y$, $z$ that it deals with is supposed to occupy an array that contains *precision* ternary digits:

```
void NSTadd(NST &x, NST &y, NST &z) {      // set z to x + y, in radix −3/2
  register int p, s, carry;
  for (p = carry = 0;  p < precision;  p++) {
    s = x[p] + y[p] + carry + 6;      // 0 ≤ s < 15 and −4 ≤ carry ≤ 4
    z[p] = s % 3;      // s mod 3                                        (3.4)
    carry = 4 − (2 ∗ (s ÷ 3));
  }
  if (carry) exit(OVERFLOW);
}
```

**4. NST integers.** A number of the form (2.1) with $l = 0$ is called an *NST integer.* We've seen that every ordinary integer is an NST integer; but many more NST integers obviously exist. (For instance, $\texttt{11} = -1/2$.) Indeed, the table above shows that the NST representation of every ordinary integer besides 0, 1, 2, $-1$, and $-3$ begins with '21'. (Otherwise the value would have a denominator that's a power of 2.)

The algorithms *NSTadvance*, *NSTnegate*, and *NSTadd* above can all be seen to work correctly with arbitrary NST integers, not only with NST numbers that happen to be actual integers.

Every NST integer is obviously a member of $\mathbb{Z}[\frac{1}{2}]$, the dyadic rationals, because it is the sum $x_0 + (-\frac{3}{2})x_1 + \cdots + (-\frac{3}{2})^u x_u$ of dyadic rationals. Conversely, every dyadic rational $p/2^n$, where $p$ is odd and $n \geq 0$, is an NST integer: For if $x = p/2^{n-1}$ is an NST integer, so is $-\frac{3}{2}x$, obtained by appending '0' at the right; and so is $x/2 = p/2^n$, obtained as the sum $-\frac{3}{2}x + x + x$ of NST integers.

Furthermore every dyadic rational has a *unique* representation as an NST integer. For if distinct NST integers had the same value, then 0 would have two representations, one of which is $(x_u \ldots x_0)_{-3/2} \neq (0\ldots0)_{-3/2}$. Shifting right, we can assume that $x_0$ is 1 or 2. But then either $-1$ or $-2$ would be an NST integer whose representation ends with $x_0 = \texttt{0}$. We could shift that representation right, getting an NST integer representation of $(-1)(-\frac{2}{3})$ or $(-2)(-\frac{2}{3})$, namely of $\frac{2}{3}$ or $\frac{4}{3}$. But that's absurd, because those numbers aren't dyadic rationals.

Thus we've proved that there's a simple one-to-one correspondence between NST integers and dyadic rationals; this is Theorem 2 in [7].

**5. Finite NST fractions.** A number of the form (2.1) with $u = x_0 = 0$ is called an *NST fraction.* In other words, it's a number that can be represented in the form

$$x \;=\; (0.x_{-1}\ldots x_l)_{-3/2} \;=\; (-\tfrac{2}{3})x_{-1} + (-\tfrac{2}{3})^2 x_{-2} + \cdots + (-\tfrac{2}{3})^{-l} x_l. \qquad (5.1)$$

Every NST number is therefore an NST integer plus an NST fraction.

NST fractions come in two forms: A *finite* NST fraction has a finite value of $l > -\infty$; an *infinite* NST fraction has infinitely many nonzero digits $x_j$.

Notice that a finite NST fraction is always a *triadic* rational, namely a number of the form $p/3^n$ where $p$ isn't a multiple of 3. For example, $.\texttt{1201} = (-17/12)/(-3/2)^3 = 34/81$ (see (2.2)).

We must keep in mind that these NST fractions are triadic rationals in the 2-adic (*not* 3-adic!) sense, as well as in the sense of ordinary real fractions. The 2-adic significance of a triadic rational number $x$ is characterized by $x^R$, as we saw above; and a gung-ho reader may verify that

$$(34/81)^R = 138197851/536870916 = (0.0100000111100101110110101100111110\ldots)_2. \qquad (5.2)$$

When we add an NST integer to a finite NST fraction, we get a "finite NST number," namely an NST number in which both $l$ and $u$ in the representation (2.1) are finite. Every finite NST number that's nonzero can be represented uniquely in the form $p/(2^m 3^n)$, where $p$ is odd and not divisible by 3. Equivalently, it can be represented uniquely in the form $P/6^n$, where $P$ is not a multiple of 6; so we can call it a *hexadic rational.*

**Theorem 5.** *There's a simple one-to-one correspondence between finite NST numbers and hexadic rationals.*

*Proof.* The representation of $p/(2^m 3^n)$ must be the representation of the dyadic rational $(-1)^n p/2^{m+n}$, shifted right by $n$ places. (The correct power of 3 is uniquely determined by the rightmost nonzero digit in the representation.)

**6. Arbitrary NST fractions.** Let $\mathcal{F}$ be the set of all NST fractions. This set is one of the key concepts studied in [7], where it is called "the tile," because every ambinumber is the sum of an NST integer (a dyadic rational) and an element of $\mathcal{F}$.

One way to get a better understanding of $\mathcal{F}$ is to study its "null slice" $\mathcal{F}_0$, namely the set of all NST fractions whose 2-adic part is zero. A bit of (possibly nontrivial) thought leads to the realization that $\mathcal{F}_0$ is precisely the set of all infinite sequences of ternary digits $x_{-1}x_{-2}\ldots$ such that

$$\bigl|(0.x_{-1}\cdots x_{-n})_{-3/2}\bigr|_2 \;<\; 1/2^n \quad \text{for all } n \geq 0. \qquad (6.1)$$

For if this norm is $\geq 1/2^n$ when $n = n'$, it will have the same value for all $n \geq n'$. (All future terms $x_{-n'-1}(-\frac{3}{2})^{-n'-1} + x_{-n'-2}(-\frac{3}{2})^{-n'-2} + \cdots$ are multiples of $2^{n'+1}$, so they won't affect $|x|_2$.)

For example, $x_{-1}$ must be 0 or 2; it can't be 1.

There are three possibilities for $x_{-1}x_{-2}$: 00, 02, 21.

There are four possibilities for $x_{-1}x_{-2}x_{-3}$: 000, 002, 021, 211.

There are six possibilities for $x_{-1}x_{-2}x_{-3}x_{-4}$: 0000, 0002, 0021, 0211, 2110, 2112.

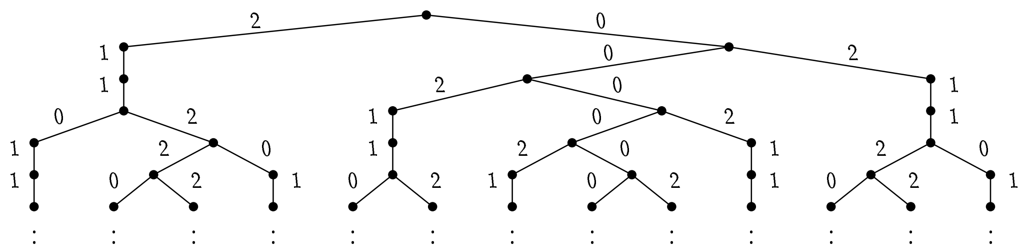
And nine possibilities for $x_{-1}x_{-2}x_{-3}x_{-4}x_{-5}$: 00000, 00002, 00021, 00211, 02110, 02112, 21101, 21120, 21122.

So what's the pattern?

Well, it's not difficult to see that each possible setting of $x_{-1} \ldots x_{-n}$ can be extended to either one or two settings of $x_{-1} \ldots x_{-n}x_{-n-1}$, because the 2-adic number $(-2/3)^{n+1}$ is congruent to $2^{n+1}$ (modulo $2^{n+2}$) for all $n \geq 0$. If $(0.x_{-1} \ldots x_{-n})_{-3/2}$ is congruent to $2^{n+1}$, we are forced to choose $x_{-n-1} = 1$, in order to satisfy (6.1). Otherwise $x_{-n-1}$ can be either 0 or 2. We never reach a dead end.

Each element of $\mathcal{F}_0$ therefore corresponds to an infinite path in the interesting tree shown here, which has either unary or binary branching at every node:

$\mathcal{T}_0$ =



(6.2)

The branches labeled '2' have been placed to the left of the branches labeled '0', on odd levels, but to the right on even levels, so that the paths of even length correspond to the natural order of numerical values from left to right. This arrangement reveals the secret of the tree: The thirteen downward paths 211011, 211220, 211222, 211201, 002110, 002112, 000021, 000000, 000002, 000211, 021120, 021122, 021101 to level 6 are precisely the NST codes for the even numbers $-14, -12, \ldots, +8, +10$ whose codes have length six or less! Indeed, condition (6.1) holds for $x_{-1} \ldots x_{-n}$ if and only if $(x_{-1} \ldots x_{-n})_{-3/2}$ is an even integer. We see also that the labels decrease by 1, mod 3, on the even levels; they *increase* by 1, mod 3, on the odd levels.

**7. The extreme paths.** The leftmost path in (6.2) begins with 211011 . . . . The rightmost path, which begins with 021101. . . , looks like it might perhaps be the same, except preceded by 0.

Let's denote the digits of the leftmost path by $a_1, a_2, \ldots$ . We can psych out the rule that governs this sequence by using the fact that $(a_1 \ldots a_n)_{-3/2}$ is an even integer, for all $n$. If we know $a_k$ for $0 \leq k < n$, let's set $b_n = (a_1 \ldots a_{n-1}0)_{-3/2}$. Then if $b_n$ is odd, we must set $a_n = 1$. Otherwise, we set $a_n = 2$ if $n$ is odd; $a_n = 0$ if $n$ is even. The recurrence is therefore

$$b_0 = 0; \quad a_n = \begin{cases} 1, & \text{if } b_n \text{ is odd;} \\ 1-(-1)^n, & \text{if } b_n \text{ is even;} \end{cases} \quad b_{n+1} = -\frac{3}{2}(b_n + a_n). \tag{7.1}$$

This rule will evaluate the first $n$ digits in order $n^2$ steps, because the number of bits in $b_n$ is of order $n$. So we quickly find that $a_1 = 2$, $a_2 = 1$, $a_3 = 1$, $a_4 = 0$, . . .; hence

$$0.21101\ 11121\ 10212\ 01020\ 11201\ 02121\ 21212\ 11010\ 11201\ 01020\ 10112\ 12011\ 20102\ 12020\ 20212\ 11120\ 11102\ 12011 \ldots \tag{7.2}$$

is the path to the smallest element of $\mathcal{F}_0$. (Is there an asymptotically faster way to compute $a_n$, given $n$, without necessarily knowing any of the digits $a_k$ for $k < n$?)

The sequence $\langle b_n \rangle$, which we've used in this computation of $\langle a_n \rangle$, begins as follows:

| $n =$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | . . . |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $b_n =$ | 0 | 0 | −3 | 3 | −6 | 9 | −15 | 21 | −33 | 48 | −75 | 111 | −168 | 252 | −381 | 570 | −858 | 1287 | −1932 | 2898 | . . . |

(7.3)

and $b_{100} = -531983820293291175$. Notice how the sign keeps changing. When $n$ is odd, $b_n$ is the largest multiple of 3 whose NST representation has $n$ digits. When $n$ is even, $b_n$ is the *smallest* multiple of 3 with that property.

We observed above that the thirteen nodes of the tree on level 6, which have the NST codes 211011, 211220, ..., 021122, 021101, correspond to the even numbers −14, −12, ..., +8, +10; those are the even numbers from $b_6 + a_6$ up to $b_5 + a_5$. Similarly, the nine nodes on level 5 have the NST codes 21101, 21122, ..., 02112, 02110; they are the even numbers +10, +8, ..., −4, −6 from $b_5 + a_5$ *down* to $b_4 + a_4$. In general, level $n$ covers the even numbers between $b_n + a_n$ and $b_{n-1} + a_{n-1}$, inclusive, when $n > 0$.

Therefore the rightmost path does indeed have the same sequence of digit labels, but preceded by '0'.

The 2-adic part of the leftmost path is 0, by definition. The real part is

$$\lambda = \textstyle\sum_{n=1}^{\infty}(-\frac{2}{3})^n a_n = -1.30849\ 63584\ 53242\ 04546\ 92827\ 89158\ 30527\ 45111+, \tag{7.4}$$

a constant that does not seem to have any known closed form. The real part of the rightmost path is, similarly,

$$-\tfrac{2}{3}\lambda = 0.87233\ 09056\ 35494\ 69697\ 95218\ 59438\ 87018\ 30074+. \tag{7.5}$$

Therefore we know enough to characterize $\mathcal{F}_0$ completely:

**Theorem 7.** *The real part of $\mathcal{F}_0$ is the closed interval*

$$[\lambda\ ..\ -\tfrac{2}{3}\lambda]. \tag{7.6}$$

*Proof.* The numerical values at every level of $\mathcal{T}_0$ are perfectly uniform.

**8. The profile of the tree for $\mathcal{F}_0$.** How many nodes does the tree (6.2) have on level $n$? This number, $w_n$, has the following values when $n$ is small:

| $n =$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | ... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $w_n =$ | 1 | 2 | 3 | 4 | 6 | 9 | 13 | 19 | 28 | 42 | 63 | 94 | 141 | 212 | 318 | 477 | 716 | 1074 | 1611 | 2417 | 3626 | 5439 | 8158 | 12237 | ... |

(8.1)

(It's another sequence that seems to have been previously unknown in the literature of mathematics.) We can compute it easily from the formulas

$$c_n = (-1)^{n+1}(b_n + a_n), \qquad w_n = 1 + (c_n + c_{n-1})/2, \qquad \text{for } n > 0, \tag{8.2}$$

because we know from the discussion above that the values on level $n$ are precisely the even numbers from $-c_n$ up to $c_{n-1}$, inclusive.

Notice that whenever $w_n$ is even, the value of $w_{n+1}$ is always $\frac{3}{2}w_n$, exactly. This is a consequence of the fact that the labels on each level of (6.2) are cyclic, with period 3; nodes that have one child strictly alternate with nodes that have two. We have in fact $|w_{n+1} - \frac{3}{2}w_n| \le \frac{1}{2}$ for all $n \ge 0$.

The values of $w_n/(\frac{3}{2})^n$ converge rapidly to the constant

$$1.09041\ 36320\ 44368\ 37122\ 44023\ 24298\ 58772\ 87593-; \tag{8.3}$$

and the formulas above imply that this constant is $-\frac{1}{2}\lambda - \frac{1}{3}\lambda = -\frac{5}{6}\lambda$.

**9. Just beyond the extremes.** From the digits of the leftmost path in (7.2), we know for example that the smallest value on level 30 is $-c_{30} =$ 21101 11121 10212 01020 11201 02121. Suppose we try to subtract 2 from that value, using the transducer (3.1): The result is

$$\texttt{21101 11121 10212 01020 11201 02121} - 2 = \texttt{21 12010 20212 01121 10111 02110 11212}, \tag{9.1}$$

a 32-digit number. That's why $c_{30}$ was the smallest value on level 30.

Similarly, if we try to add 2 to the largest value, $c_{29}$, on level 30, we get

$$\texttt{02110 11112 11021 20102 01120 10212} + 2 = \texttt{2 11201 02021 20112 11011 10211 01121}, \tag{9.2}$$

a 31-digit number.

One way to understand what's going on here is to notice the sequence of identities

$$2 = 21\bar1 = 21\bar11\bar1 = 21\bar11\bar11\bar1 = 21\bar11\bar11\bar11\bar1 = 21\bar11\bar11\bar11\bar11\bar1 = 21\bar11\bar11\bar11\bar11\bar11\bar1 = 21\bar11\bar11\bar11\bar11\bar11\bar11\bar1 = \cdots, \tag{9.3}$$

where we allow ourselves to use the digit $\bar1 = -1$. (Also $-2 = 21 = 21\bar11 = 21\bar11\bar11 = \cdots$.) For example, we get the right-hand side of (9.2) by plugging in $2\ 1\bar11\bar11\ \bar11\bar11\bar1\ 1\bar11\bar11\ \bar11\bar11\bar1\ 1\bar11\bar11\ \bar11\bar11\bar1$ for the '+2' on the left. These identities are readily proved by induction. And this trick works because, by definition (7.1), $a_n \ne 0$ when $n$ is odd, $a_n \ne 2$ when $n$ is even; thus $a_n + (-1)^n$ is always a valid digit.

We've now proved that $(\lambda, 0)$, the left endpoint of $\mathcal{F}_0$, has two different representations as an NST number:

$$\begin{aligned} &\texttt{0.21101 11121 10212 01020 11201 02121 21212 11010} \;\ldots \\ &= \texttt{21.12010 20212 01121 10111 02110 11212 12121 20101} \;\ldots\;. \end{aligned} \tag{9.4}$$

For if $\delta_n$ is the difference between the $n$-digit approximations of those numbers, we have $|\delta_n| = 2^{n+1}/3^n$ in the real part and $|\delta_n|_2 = 1/2^{n+1}$ in the 2-adic part.

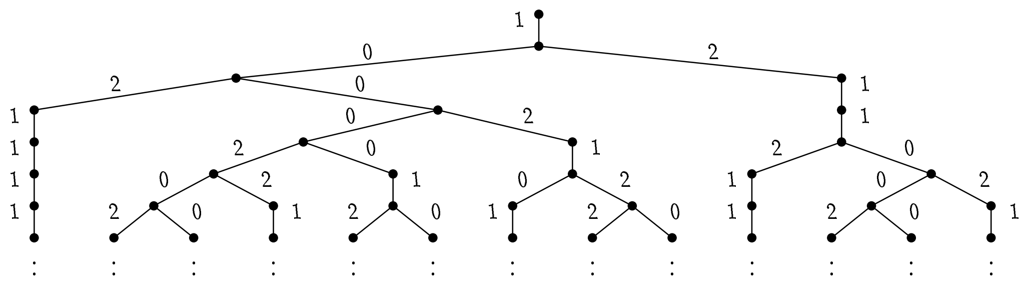
Notice that the fraction $\texttt{0.12010 20212 01121 10111 02110 11212 12121 20101} \ldots$ is a member of $\mathcal{F}$. Its 2-adic part must be equal to 2, because $\texttt{21} = -2$. Thus, it's an element of $\mathcal{F}_2$. Indeed, it's the element of $\mathcal{F}_2$ with the largest real part.

Multiplying (9.4) by $-2/3$, which amounts to shifting right one place, shows us in a similar way that the fraction $\texttt{0.11201 02021} \ldots$ is the member of $\mathcal{F}_{-2}$ whose real part is smallest.

**10. Arbitrary slices of $\mathcal{F}$.** The 2-adic part, $q$, of every element of the tile $\mathcal{F}$ is *even*; that is, it's twice a 2-adic integer; because $(-2/3)^n$ is even for all $n \geq 1$. Equivalently, $|q|_2 < 1$.

Let $\mathcal{F}_q$ be the set of all tile elements whose 2-adic part equals a given 2-adic number $q$. It turns out that the elements of $\mathcal{F}_q$ can always be characterized as the infinite paths of a tree $\mathcal{T}_q$, which is analogous to the tree $\mathcal{T}_0$ depicted in (6.2).

For example (catch-22?), here's the tree $\mathcal{T}_{-22}$:




(10.1)

As before, every node of this tree has either one or two children, strictly alternating between one and two on each level. And again the labels change cyclically (mod 3), moving $\ldots\texttt{012012}\ldots$ on the odd levels and $\ldots\texttt{210210}\ldots$ on the even levels.

The leftmost path of $\mathcal{T}_q$ can be computed with a procedure analogous to the recurrence (7.1) that we used for $\mathcal{T}_0$, but it's somewhat more elaborate:

$$B_0 = 0; \quad a_n = \begin{cases} 1, & \text{if } B_n \not\equiv (-3)^n q \text{ (modulo } 2^{n+1}); \\ 1-(-1)^n, & \text{otherwise;} \end{cases} \quad B_n = -3(B_{n-1} + 2^{n-1}a_{n-1}). \tag{10.2}$$

Here $B_n = 2^n(a_1 \ldots a_{n-1}0)_{-3/2}$, and the general condition

$$B_n + 2^n a_n \equiv (-3)^n \quad \text{modulo } 2^{n+1} \tag{10.3}$$

replaces (6.1).

With this new recurrence, the auxiliary integers $|B_n|$ grow as $3^n$, not $(3/2)^n$ as before. However, if our goal is to compute only the most significant digits $a_1, \ldots, a_n$, it suffices to know the values of $B_1, \ldots, B_n$ modulo $2^{n+1}$; hence we needn't maintain full precision. The running time to compute the first $n$ steps of the leftmost path remains $O(n^2)$, even if we compute the $B$'s exactly.

Let's say that an *alt-unary-binary tree* is an ordered tree that has the properties just mentioned for $\mathcal{T}_q$, possibly truncated at a certain level: (i) Every nonleaf node has either one or two children. (ii) No adjacent nodes on the same level have the same degree. (iii) All leaves, if any, occur on the same level. (The name "alternating unary-binary tree" has incidentally been used for an entirely different concept [1].)

Any infinite alt-unary-binary tree can be given labels in a unique way that will make it the same as $\mathcal{T}_q$, for some even 2-adic integer $q$. (The branch below a unary node is always labeled '$\texttt{1}$'; the branches below a binary node are always labeled '$\texttt{2}$' and '$\texttt{0}$' if they go to an odd level, otherwise they're labeled '$\texttt{0}$' and '$\texttt{2}$'.) In particular, the shape of the tree is enough to tell us all of the labels $a_1a_2\ldots$ of its leftmost path.

Conversely, the labels of the leftmost path, of any alt-unary-binary tree, uniquely determine the shape of the entire tree! For example, (10.1) shows that the leftmost path of $\mathcal{T}_{-22}$ begins with $a_1 \ldots a_7 = \texttt{1021111}$, and that level 7 has 13 nodes. Assume that we've already shown how to reconstruct those levels when given only $a_1 \ldots a_7$. The next value $a_8$ will be either `0` or `1`. If it's `0`, level 8 will have 20 nodes, and the branches to them will be labeled `02102102102102102102`; we'll know exactly how to connect each of those 20 nodes to its parent. And if it's `1`, level 8 will have 19 nodes, and the label sequence will be `1021021021021021021`. (Notice that, in either case, we'll be able to draw the nodes so that they're equally spaced, just 2/3 of the distance that separates the nodes on level 7 — thereby continuing the tradition that has been adopted in this diagram as well as in (6.2).)*

We've now established a simple one-to-one correspondence between infinite alt-unary-binary trees and the even 2-adic numbers. The correspondence was made via the leftmost path in the tree $\mathcal{T}_q$ for $q$ (which corresponds, in turn, to the smallest real part in $\mathcal{F}_q$).

We could equally well have considered the *rightmost* path (which corresponds, in turn, to the *largest* real part in $\mathcal{F}_q$). The rightmost path is obtained by a rule almost identical to (10.2): We merely need to change '$1-(-1)^n$' to '$1+(-1)^n$' in the definition of $a_n$ when $n > 0$; $a_0$ should remain 0.

All paths between the leftmost and the rightmost belong to $\mathcal{T}_q$; and the real parts of their adjacent nodes at each level $k > 0$ differ by exactly $2^{k+1}/3^k$. Therefore the closed-interval property we observed for $q = 0$ is true in general:

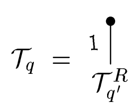
**Theorem 10.** *Let $a_1 a_2 \ldots$ and $a'_1 a'_2 \ldots$ be the leftmost and rightmost paths of $\mathcal{T}_q$, where $q$ is an arbitrary even 2-adic number. Then the real part of $\mathcal{F}_q$ is the closed interval*

$$[\lambda^{(q)} \,..\, \rho^{(q)}], \quad \text{where } \lambda^{(q)} = (0.a_1 a_2 \ldots)_{-3/2} \text{ and } \rho^{(q)} = (0.a'_1 a'_2 \ldots)_{-3/2}. \tag{10.4}$$

The left endpoint $\lambda^{(0)} = \lambda$ was evaluated in (7.4) above, and the right endpoint $\rho^{(0)} = -\frac{2}{3}\lambda$ is in (7.5).

**11. Neighboring slices.** When we looked beyond the extreme points of $\mathcal{F}_0$, we noticed that its smallest point $(\lambda, 0)$ was also a member of $\mathcal{F}_2 + 2$, and that its largest point $(-\frac{2}{3}\lambda, 0)$ was also a member of $\mathcal{F}_{-2} + 2$. In general it turns out that the smallest and largest points of $\mathcal{F}_q$ will belong respectively to $\mathcal{F}_{q+2} + 2$ and to $\mathcal{F}_{q-2} + 2$, when the even 2-adic number $q$ is *arbitrary.*

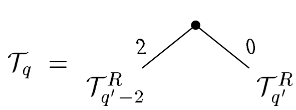
This phenomenon is a consequence of a simple recursive structure that underlies the entire family of trees $\mathcal{T}_q$. According to (10.2), the first digit $a_1$ of the leftmost path is equal to '`1`' when $q$ is "singly even" (that is, when $q \bmod 4 = 2$); and it's '`2`' when $q$ is "doubly even" (that is, when $q \bmod 4 = 0$). In the first case, it's not difficult to see that




$$\text{for } q' = -\tfrac{3}{2}q - 1, \tag{11.1}$$

where $\mathcal{T}^R_{q'}$ is the mirror image of $\mathcal{T}_{q'}$ under left-right reflection. The leftmost path of $\mathcal{T}_q$ is the same as the rightmost path of $\mathcal{T}_{q'}$, preceded by '`1`'. And the rightmost path of $\mathcal{T}_q$ is the same as the leftmost path of $\mathcal{T}_{q'}$, preceded by '`1`'.

A similar law defines the two subtrees of $\mathcal{T}_q$ in the second case:

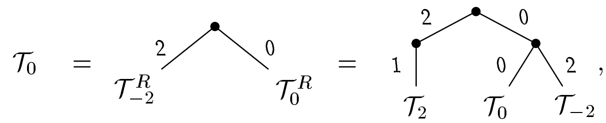



$$\text{for } q' = -\tfrac{3}{2}q. \tag{11.2}$$

The leftmost path of $\mathcal{T}_q$ is the same as the rightmost path of $\mathcal{T}_{q'-2}$, preceded by '`2`'; the rightmost path of $\mathcal{T}_q$ is the same as the leftmost path of $\mathcal{T}_{q'}$, preceded by '`0`'. And when the leftmost path of $\mathcal{T}_{q'-2}$ is preceded by '`2`', we get the "right neighbor" of the path obtained by prefixing '`0`' to the rightmost path of $\mathcal{T}_{q'}$, namely a path whose nodes are neighbors at every level.

* By the way, it turns out that $a_8 a_9 \ldots = \texttt{010211110212} \ldots$ .

For example, when we apply these rules starting with $q = 0$, we get



(11.3)

which nicely describes the inherent recursive structure apparent in (6.2).

And if we continue to develop that tree, we find that the 9 subtrees on level 5 of (6.2) are respectively

$$\mathcal{T}^R_{-10},\ \mathcal{T}^R_{-8},\ \mathcal{T}^R_{-6},\ \mathcal{T}^R_{-4},\ \mathcal{T}^R_{-2},\ \mathcal{T}^R_{0},\ \mathcal{T}^R_{2},\ \mathcal{T}^R_{4},\ \mathcal{T}^R_{6}\ ; \tag{11.4}$$

thus the 13 subtrees indicated by '$\vdots$' in that diagram are respectively

$$\mathcal{T}_{14}, \mathcal{T}_{12}, \mathcal{T}_{10}, \mathcal{T}_{8}, \mathcal{T}_{6}, \mathcal{T}_{4}, \mathcal{T}_{2}, \mathcal{T}_{0}, \mathcal{T}_{-2}, \mathcal{T}_{-4}, \mathcal{T}_{-6}, \mathcal{T}_{-8}, \mathcal{T}_{-10}. \tag{11.5}$$

**12. The narrowest and widest slices.** Let

$$w^{(q)} = (\rho^{(q)} - \lambda^{(q)})/2 \tag{12.1}$$

denote the width of the slice $\mathcal{F}_q$. This number is the limit of $\left(\frac{2}{3}\right)^n w_n^{(q)}$ as $n \to \infty$, where $w_n^{(q)}$ is the number of nodes on level $n$ of $\mathcal{T}_q$. For example, we found the first terms of the sequence $w_n^{(0)}$ in (8.1) above.

The profile of $\langle w_n^{(q)} \rangle$ of any alt-unary-binary tree clearly satisfies

$$w_{n+1}^{(q)} \;=\; \left\lfloor \tfrac{3}{2} w_n^{(q)} \right\rfloor \qquad \text{or} \qquad \left\lceil \tfrac{3}{2} w_n^{(q)} \right\rceil, \tag{12.2}$$

because either $\lfloor w_n^{(q)}/2 \rfloor$ or $\lceil w_n^{(q)}/2 \rceil$ of the nodes on level $n$ will add to the total of nodes on level $n+1$ by having two children.

When $q$ is singly even, $\mathcal{T}_q$ has the form (11.1); hence $w_1^{(q)} = 1$, and $w_n^{(q)} = w_{n-1}^{(q')}$ for $n > 1$. Consequently $w^{(q)} = \frac{2}{3} w^{(q')}$ in this case. In fact, we have

$$\lambda^{(q)} = -\tfrac{2}{3}(1 + \rho^{(q')}); \quad \rho^{(q)} = -\tfrac{2}{3}(1 + \lambda^{(q')}); \quad q' = -\tfrac{3}{2}q - 1. \tag{12.3}$$

When $q$ is doubly even, $\mathcal{T}_q$ has the form (11.2) and we have

$$w^{(q)} = \tfrac{2}{3}(w^{(q'-2)} + w^{(q')}); \quad \lambda^{(q)} = -\tfrac{2}{3}(2 + \rho^{(q'-2)}); \quad \rho^{(q)} = -\tfrac{2}{3}\rho^{(q')}. \tag{12.4}$$

There seems to be no simple formula for $\rho^{(q')}$. Equivalently, there seems to be no simple formula for $\lambda^{(q'-2)}$, because we always have

$$\lambda^{(q-2)} \;=\; \rho^{(q)} - 2. \tag{12.5}$$

Incidentally, I happened to notice a curious thing when exploring these trees: The profile of $\mathcal{T}_{-2}$ is

$$\begin{array}{r@{\,}l} n &= 0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ \ 8\ \ \ 9\ \ 10\ 11\ 12\ \ 13\ \ \ 14\ \ \ 15\ \ \ 16\ \ \ 17\ \ \ 18\ \ \ \ 19\ \ \ \ \ 20\ \ \ \ \ 21\ \ \ \ \ 22\ \ \ \ \ 23\ \ldots \\ w_n^{(-2)} &= 1\ 1\ 1\ 2\ 3\ 4\ 6\ 9\ 14\ 21\ 31\ 47\ 71\ 106\ 159\ 239\ 358\ 537\ 806\ 1209\ 1813\ 2719\ 4079\ 6119\ \ldots \end{array} \tag{12.6}$$

and the first eleven terms of this sequence turn out to be a perfect match to three of the sequences in the OEIS [6] — namely A073941, A078620, and A143951. But the next three terms of those sequences are respectively $(47, 70, 105)$, $(46, 70, 106)$, $(47, 71, 107)$, thus *not* a match to the terms $(47, 71, 106)$ of (12.6). Close but no cigar. In fact, (11.3) gives us a remarkable relation between the profiles of $\mathcal{T}_{-2}$ and $\mathcal{T}_0$:

$$w_n^{(-2)} \;=\; w_{n+1}^{(0)} - w_n^{(0)}, \qquad \text{for all } n \ge 0. \tag{12.7}$$

According to (12.2), the *narrowest* slice occurs when $w^{(q)}_{n+1} = \lfloor \frac{3}{2} w^{(q)}_n \rfloor$ for all $n$, namely when $w^{(q)}_n = 1$ for all $n \ge 0$ and the tree has just one path! That case clearly occurs if and only if $q = q'$ in (11.1), if and only if

$$q = \mathtt{0.111111111}\ldots = \textstyle\sum_{n=1}^{\infty} \bigl(-\frac{2}{3}\bigr)^n = -\frac{2}{5}; \quad \text{thus } w^{(-2/5)} = 0. \tag{12.8}$$

The narrowest slice when $q \bmod 4 = 0$ occurs when $w^{(q)}_1 = 2$ and $w^{(q)}_{n+1} = \lfloor \frac{3}{2} w^{(q)}_n \rfloor$ for all $n > 1$. Then

$$\begin{array}{r} n = 0\ 1\ 2\ 3\ 4\ 5\ 6\ \ 7\ \ 8\ \ 9\ \ 10\ 11\ \ 12\ \ 13\ \ 14\ \ 15\ \ 16\ \ 17\ \ 18\ \ 19\ \ 20\ \ 21\ \ 22\ \ldots \\ w^{(q)}_n = 1\ 2\ 3\ 4\ 6\ 9\ 13\ 19\ 28\ 42\ 63\ 94\ 141\ 211\ 316\ 474\ 711\ 1066\ 1599\ 2398\ 3597\ 5395\ 8092\ \ldots \end{array} \tag{12.9}$$

and this is OEIS sequence A061418 if we leave out the initial ‘1’ [6]. It’s only slightly smaller than the profile of $\mathcal{T}_0$ in (8.1), differing first when $n = 13$.

The simplest way to obtain the profile (12.9) is to let $\mathcal{T}_q$ be the tree whose leftmost path is $\mathtt{21111111}\ldots$. For in that case, the nodes at the left and right of each level with $w^{(q)}_n$ odd will be unary. And to get that case, we can set $q' - 2 = -2/5$ in (11.2); hence $q = -\frac{2}{3} q' = -\frac{16}{15}$. (Another way to compute this value is to note that $q = \mathtt{0.2111111}\ldots = -\frac{2}{5} - \frac{2}{3}$.) We have proved that the minimum width $w^{(q)}$ when $q \bmod 4 = 0$ is

$$w^{(-16/15)} = \lim_{n\to\infty} \bigl(\tfrac{2}{3}\bigr)^n w^{(-16/15)}_n = 1.08151\ 36685\ 89844\ 87730\ 46339\ 88599\ 54940\ 87107+. \tag{12.10}$$

Infinitely many doubly even 2-adic numbers $q$ will have this minimum width. Indeed, the relevant $q$ with smallest real part is obtained when the $n$th step of its leftmost path is ‘$\mathtt{1}$’ if $w^{(q)}_n$ is odd in (12.9), otherwise ‘$\mathtt{2}$’ if $n$ is odd, otherwise ‘$\mathtt{0}$’:

$$\mathtt{0.20102\ 11120\ 10112\ 01010\ 11201\ 01121\ 21111\ 12010\ 11111\ 11010\ 20201\ 12010\ 10212\ 12110\ 11202\ 01021\ 20202}\ \ldots. \tag{12.11}$$

In this case the *rightmost* path will be $\mathtt{0.01111111}\ldots$ (!); so this value of $q$ is $-\frac{2}{5} + \frac{2}{3} = \frac{4}{15}$.

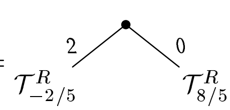
To get the set of *all* doubly even $q$ whose width is minimum,* we simply change any subset of the non-$\mathtt{1}$ digits of (12.11) to ‘$\mathtt{1}$’, except that the very first digit must remain ‘$\mathtt{2}$’.

OK, we’ve settled the question of the narrowest slices; what are the *widest* ones? In this case we must have $w^{(q)}_{n+1} = \lceil \frac{3}{2} w^{(q)}_n \rceil$ for all $n \ge 0$:

$$\begin{array}{r} n = 0\ 1\ 2\ 3\ 4\ \ 5\ \ 6\ \ 7\ \ 8\ \ 9\ \ 10\ \ 11\ \ 12\ \ 13\ \ 14\ \ 15\ \ 16\ \ 17\ \ 18\ \ 19\ \ 20\ \ 21\ \ 22\ \ldots \\ w^{(q)}_n = 1\ 2\ 3\ 5\ 8\ 12\ 18\ 27\ 41\ 62\ 93\ 140\ 210\ 315\ 473\ 710\ 1065\ 1598\ 2397\ 3596\ 5394\ 8091\ 12137\ \ldots \end{array} \tag{12.12}$$

It’s another famous sequence (OEIS A061419). And guess what? $w^{(q)}_n$ in (12.12) is exactly 1 less than the value of $w^{(q)}_{n+1}$ in (12.9)!

That fact can easily be proved by induction. But there’s also a nice way to connect it to alt-unary-binary trees, thereby learning more. For it’s easy to see that we get the profile (12.12) when the rightmost path of $\mathcal{T}_q$ *never* contains the digit ‘$\mathtt{1}$’. In that case an odd number on level $n$ will always be rounded upwards when we calculate $\lceil \frac{3}{2} w^{(q)}_n \rceil$. Therefore the maximum width is obtained when $q = \mathtt{0.02020202}\ldots = \frac{8}{9}/(1 - \frac{4}{9}) = \frac{8}{5}$. And the point is that one of the special cases of (11.2) is

$\mathcal{T}_{-16/15} =$ 

(12.13)

and we already *know* both $\mathcal{T}_{-16/15}$ and $\mathcal{T}_{-2/5}$! This representation of $\mathcal{T}_{-16/15}$, which we’ve seen is a tree with minimum width, makes it clear that $\mathcal{T}_{-16/15}$ has exactly one more node on level $n+1$ than $\mathcal{T}_{8/5}$ has on level $n$, and that the maximum slice width is

$$w^{(8/5)} = \tfrac{3}{2} w^{(-16/15)} = 1.62227\ 05028\ 84767\ 31595\ 69509\ 82899\ 32411\ 30661+. \tag{12.14}$$

This is a famous constant, which Odlyzko and Wilf called $K(3)$ in their analysis of the Josephus problem [5].

* One can show without difficulty that if $w^{(q)}_{n+1} > \lfloor \frac{3}{2} q \rfloor$ for any $n \ge 1$, the error will eventually be magnified and the final limit will be greater than (12.10).

As before, infinitely many 2-adic numbers $q$ have this minimum width. We can get them all by starting with

$$\texttt{0.21201 11021 21102 12121 10212 11010 11111 02121 11111 12120 20211 02121 20101 01121 10202 12010 20202} \ \ldots, \tag{12.15}$$

which is the *leftmost* path of $\mathcal{T}_{8/5}$. (It is obtained with the same idea that we used in (12.11), but now we choose the digit '1' wherever $w_n^{(q)}$ in (12.12) is *odd*.) Then we change any subset of the 1's into either '0' or '2', according to the parity of the step number. For example, if we make all possible changes, we get 20202020..., which is the leftmost path of $\mathcal{T}_{-12/5}$.

The 13 subtrees in (11.5) have 13 different widths, varying in a somewhat strange pattern:

$$\begin{array}{rccccccccccccc} q = & 14 & 12 & 10 & 8 & 6 & 4 & 2 & 0 & -2 & -4 & -6 & -8 & -10 \\ w^{(q)} \approx & 0.5231 & 1.2513 & 0.8111 & 1.5547 & 0.2325 & 1.6076 & 0.8178 & 1.0904 & 0.5452 & 1.2267 & 1.0365 & 1.3749 & 0.3487 \end{array} \tag{12.16}$$

These values represent the relative numbers of nodes that will eventually be descended from those subtrees $\mathcal{T}_{14}, \mathcal{T}_{12}, \ldots, \mathcal{T}_{-10}$ in the infinite tree $\mathcal{T}_0$:

$$\mathcal{T}_{14} \ \ \mathcal{T}_{12} \ \ \mathcal{T}_{10} \ \ \mathcal{T}_{8} \ \ \mathcal{T}_{6} \ \ \mathcal{T}_{4} \ \ \mathcal{T}_{2} \ \ \mathcal{T}_{0} \ \ \mathcal{T}_{-2} \ \ \mathcal{T}_{-4} \ \ \mathcal{T}_{-6} \ \ \mathcal{T}_{-8} \ \ \mathcal{T}_{-10} \tag{12.17}$$

**13. Enumeration of alt-unary-binary trees.** Let $A(n,w,l)$ be the number of alt-unary-binary trees that have exactly $n$ nodes, exactly $w$ of which are leaves at level $l$. This quantity can be computed from the following readily devised recurrence:

$$A(n,w,l) = \begin{cases} 0, & \text{if } n \le 0 \text{ or } w \le 0 \text{ or } l < 0; \\ 0, & \text{if } n = 1 \text{ and } (w \ne 1 \text{ or } l \ne 0); \\ 1, & \text{if } n = 1 \text{ and } w = 1 \text{ and } l = 0; \\ A(n-w, 2\lfloor w/3 \rfloor + 1, l-1), & \text{if } n > 1 \text{ and } w \bmod 3 \ne 0; \\ 2A(n-w, 2w/3, l-1), & \text{if } n > 1 \text{ and } w \bmod 3 = 0. \end{cases} \tag{13.1}$$

The total number of alt-unary-binary trees whose leaves all belong to level $l$ is then

$$A(n,l) = \sum_{w=1}^{\infty} A(n,w,l); \tag{13.2}$$

this sum is finite because $A(n,w,l) = 0$ whenever $w$ exceeds the value $w_l^{(q)}$ in (12.9).

Therefore the total number of alt-unary-binary trees with $n$ nodes is

$$A(n) = \sum_{l=0}^{n} A(n,l), \tag{13.3}$$

and we have the following values for small $n$:

$$\begin{array}{r} n = 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16\ 17\ 18\ 19\ 20\ 21\ 22\ 23\ 24\ 25\ 26\ 27\ 28\ 29\ 30\ 31\ 32\ \ldots \\ A(n) = 1\ 1\ 2\ 2\ 2\ 4\ 4\ 4\ 4\ \ 6\ \ \ 8\ \ \ 8\ \ \ 8\ \ \ 8\ \ \ 8\ \ \ 8\ 12\ 12\ 14\ 16\ 16\ 16\ 16\ 16\ 16\ 24\ 24\ 24\ 26\ 28\ 28\ 32\ 32\ \ldots \end{array} \tag{13.4}$$

with $A(n) = 32$ for $31 \le n \le 37$ and $A(38) = 40$.

**14. Double points.** We observed in (9.4) that NST representations are not always unique. In fact, the left endpoint

$$\lambda^{(q)} = (0.a_1 a_2 a_3 a_4 a_5 a_6 \ldots)_{-3/2} \tag{14.1}$$

of any slice $\mathcal{F}_q$ always has another representation

$$\lambda^{(q)} = (21.(a_1{-}1)(a_2{+}1)(a_3{-}1)(a_4{+}1)(a_5{-}1)(a_6{+}1)\ldots)_{-3/2} = \rho^{(q+2)} - 2. \tag{14.2}$$

The basic reason is that the digits of a left endpoint always satisfy the "even-odd property":

$$a_{2n-1} \ne 0 \text{ and } a_{2n} \ne 2, \text{ for all } n \ge 1; \tag{14.3}$$

furthermore (9.3) gives us the identity

$$21.\bar{1}1\bar{1}1\bar{1}1\bar{1}1\bar{1}1\bar{1}1\bar{1}1\bar{1}1\bar{1}1\bar{1}1 \ldots = 0. \tag{14.4}$$

Therefore we can add (14.4) to (14.1), getting (14.2).

Similarly, the right endpoint of any slice always has two representations

$$\rho^{(q)} = (0.a_1'a_2'a_3'a_4'a_5'a_6'\ldots)_{-3/2} = (2.(a_1'{+}1)(a_2'{-}1)(a_3'{+}1)(a_4'{-}1)(a_5'{+}1)(a_6'{-}1)\ldots)_{-3/2} = \lambda^{(q-2)}+2, \quad (14.5)$$

because the digits of a right endpoint always satisfy the "odd-even property"

$$a_{2n-1}' \neq 2 \text{ and } a_{2n}' \neq 0, \text{ for all } n \geq 1, \quad (14.6)$$

and because

$$2.1\bar{1}1\bar{1}1\bar{1}1\bar{1}1\bar{1}1\bar{1}1\bar{1}1\bar{1}1\bar{1}1\bar{1}\ldots = 0. \quad (14.7)$$

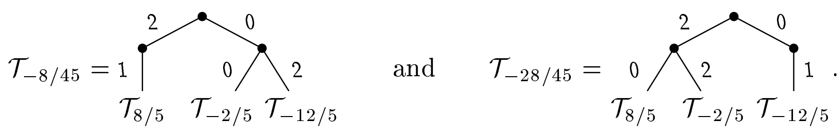
Conversely, every sequence of digits that satisfies the even-odd property is the left endpoint of $\mathcal{F}_q$ for the $q$ that it defines; and every sequence with the odd-even property is a right endpoint. So (14.2) and (14.5) are essentially instances of the same phenomenon, but with the left and right sides of the equations swapped.

The trees $\mathcal{T}_q$ give us an intuitive picture of this situation: Whenever a two-way branch occurs in such a tree, the rightmost path below the left branch and the leftmost path below the right branch will converge to the same final point at the very "bottom" (at level $\infty$).

The fraction $-\frac{2}{5} = 0.111111111\ldots$ is special, because its digits contain neither '0' nor '2'. Therefore it has both the even-odd property and the odd-even property, giving it *three* different NST representations(!):

$$21.0202020202\ldots \;=\; 0.1111111111\ldots \;=\; 2.2020202020\ldots. \quad (14.8)$$

Sometimes the trivial tree $\mathcal{T}_{-2/5}$ occurs as a subtree of a larger one. That happens, for example in




(14.9)

At the bottom of such trees we'll have a triple point.

It's natural to conjecture that *all* instances of double NST representation arise in essentially the same way, by shifting the radix point in instances such as (14.5). In other words, we always get from one ambiguous representation to the other by adding or subtracting a suitably shifted version of zero, as represented in (14.7). That's almost true, but not quite: To get from $21.0202020202\ldots$ to $2.2020202020\ldots$ in (14.8), we need to add (14.7) *twice*. (It's the unique case where $\rho^{(q)} = \lambda^{(q-4)}$.) We shall now prove a suitably modified conjecture.

**Theorem 14.** *The ambinumber $x$ has more than one NST representation if and only if $x = \left(-\frac{3}{2}\right)^m(r + \lambda^{(q)})$ for some integer $m$, some dyadic rational $r$, and some 2-adic $q$.*

*Proof.* Suppose that two NST representations

$$(x_u x_{u-1}\ldots x_0.x_{-1}\ldots)_{-3/2} = \sum_{n\leq u}\left(-\frac{3}{2}\right)^n x_n \quad\text{and}\quad (x_u'x_{u-1}'\ldots x_0'.x_{-1}'\ldots)_{-3/2} = \sum_{n\leq u}\left(-\frac{3}{2}\right)^n x_n' \quad (14.10)$$

both define the same ambinumber. We can assume that $x_u \neq 0$, after multiplying both representations by a suitable power of $(-3/2)$. (If all $x_n$ are zero, interchange $x_n$ and $x_n'$.) We can also assume that $x_u \neq x_u'$; otherwise we could subtract $(-3/2)^u x_u$ from both representations and renormalize. Finally, after both assumptions hold, we shall change notation, for convenience, writing $a_n$ instead of $x_{-n}$ and $a_n'$ instead of $x_{-n}'$. That leaves us with

$$(a_0.a_1a_2a_3a_4\ldots)_{-3/2} = (a_0'.a_1'a_2'a_3'a_4'\ldots)_{-3/2}, \qquad a_0 \neq 0, \qquad a_0 > a_0'. \quad (14.11)$$

All of these operations preserve the form of $x$ that was stated in the theorem.

The fraction parts $(.a_1a_2\ldots)_{-3/2}$ and $(.a_1'a_2'\ldots)_{-3/2}$ in (14.11) are 2-adically even. Therefore $a_0 - a_0'$ is even, and the only possibility is $(a_0, a_0') = (2, 0)$:

$$(2.a_1a_2a_3a_4\ldots)_{-3/2} = (0.a_1'a_2'a_3'a_4'\ldots)_{-3/2}. \quad (14.12)$$

We shall prove that either (i) the digits $a_n$ satisfy the even-odd property, or (ii) the digits $a_n'$ satisfy the odd-even property, or (iii) both.

The first important observation is that, modulo 4, we have $2-\frac{2}{3}a_1 \equiv -\frac{2}{3}a'_2$; hence $6-2a_1 \equiv -2a'_2$. Thus

$$a_1 \equiv a'_1 + 1 \quad (\text{modulo } 2). \tag{14.13}$$

*Case 1.* If $a_1 = a'_1 + 1$, we can multiply by $-3/2$ and assume that $a'_1 = 0$:

$$(21.a_2a_3a_4\ldots)_{-3/2} = (0.a'_2a'_3a'_4\ldots)_{-3/2}. \tag{14.14}$$

Now add 2 to both sides, getting

$$(0.a_2a_3a_4\ldots)_{-3/2} = (2.a'_2a'_3a'_4\ldots)_{-3/2}. \tag{14.15}$$

This has the form of (14.12), but with sides reversed and subscripts increased by 1. So we can repeat this process until Case 1 no longer holds, or until condition (iii) has been verified.

*Case 2.* If $a'_1 = a_1 + 1$, we can multiply by $-3/2$ and assume that $a_1 = 0$:

$$(20.a_2a_3a_4\ldots)_{-3/2} = (1.a'_2a'_3a'_4\ldots)_{-3/2}. \tag{14.16}$$

The largest possible value on the left is $20.0202020202\ldots = -3 + \frac{8}{5} = -\frac{7}{5}$. The smallest possible value on the right is $1.2020202020\ldots = 1 - \frac{12}{5} = -\frac{7}{5}$. Therefore we must have $a_2 = a_4 = \cdots = a'_3 = a'_5 = \cdots = 0$ and $a_3 = a_5 = \cdots = a'_2 = a'_4 = \cdots = 2$; condition (ii) holds. QED.

**Corollary 14.** *An NST representation (2.1) is unique if and only if either (i) $x_{2n}$ is '0' for infinitely many $n$ and $x_{2n+1}$ is '0' for infinitely many $n$ or (ii) $x_{2n}$ is '2' for infinitely many $n$ and $x_{2n+1}$ is '2' for infinitely many $n$.*

*Proof.* Those are precisely the cases when none of the powers $\left(-\frac{2}{3}\right)^m x$ are equal to $r + \lambda^{(q)}$ for any dyadic rational $r$ and 2-adic $q$.

**15. The unit elements (1, 0) and (0, 1).** The ambinumber $(1,0)$ isn't in $\mathcal{F}_0$, according to (7.5). However, the negative of that ambinumber, $(-1,0)$, does belong to $\mathcal{F}_0$, according to (7.5). Let's therefore compute the NST representation of $(-1,0)$, which is a path in the tree (6.2).

The path 211220, which corresponds to the fraction 0.211220, has real part $-12\cdot(-\frac{2}{3})^6 = -768/729$, because $211220 = -12$. That's less than $-1$. The next path to the right is 211222, which corresponds to a real part $-10\cdot(-\frac{2}{3})^6 = -640/729$ in a similar fashion; and that's *greater* than $-1$. So the infinite path we seek must begin with 21222.

The value of the next digit — the next step in the path — isn't so clear, however. At level 15 of the tree, $-1$ lies between $438\cdot(2/3)^{15}$ and $436\cdot(-2/3)^{15}$; and we have $438 = 211220011212120$, $436 = 211222102121211$. The true value isn't revealed until we get to level 16. Eventually, however, we learn that

$$(-1,0) = 0.21122\ 00112\ 12120\ 10020\ 20121\ 22002\ 02000\ 12212\ 01200\ 02020\ 22022\ 22101\ 22200\ 21011\ 11011\ \ldots\,. \tag{15.1}$$

Consequently, adding $(1,1)$,

$$(0,1) = 1.21122\ 00112\ 12120\ 10020\ 20121\ 22002\ 02000\ 12212\ 01200\ 02020\ 22022\ 22101\ 22200\ 21011\ 11011\ \ldots\,. \tag{15.2}$$

And the negation of (15.1), using (3.3), gives us the other unit element:

$$(1,0) = 2.11201\ 02211\ 11112\ 20212\ 12202\ 01021\ 21002\ 20111\ 01100\ 21210\ 01210\ 10201\ 10100\ 02012\ 12012\ \ldots\,. \tag{15.3}$$

(The validity of the rightmost digits that are shown in (15.3) depends on some of the digits that were suppressed in (15.1).)

There are many other ways to compute $(1,0)$. For example, we can readily find $(-\frac{2}{3},0) = 0.21120\ 10221\ \ldots$ in $\mathcal{T}_0$, then shift left 1.

It's possible to multiply one ambinumber by another, using radix $-3/2$ arithmetic in essentially the normal way (because we know how to add and shift). Thus, for example, we can obtain the ambinumber $(x,y)$ for any dyadic rationals $x$ and $y$ by using the rule

$$(x,y) \;=\; x\cdot(1,0)+y\cdot(0,1), \tag{15.4}$$

where $(1,0)$ and $(0,1)$ are given by (15.3) and (15.2). (The answer will be obtained as a limit, of course.)

To test all this theory out, the reader is invited to multiply $(1,0)$ by $(0,1)$, at least to low precision, thus verifying that the result does indeed lie within a very small neighborhood of $(0,0)$. For example,

$$\texttt{1.21}\times\texttt{2.11}=\texttt{2.11}+\texttt{0.422}+\texttt{0.0211}=\texttt{2.5531}=\texttt{0.2531}=\texttt{0.0231}=\texttt{0.0001}. \tag{15.5}$$

**16. OK, so what about pi?** Anybody who knows me will realize that I couldn't end this discussion without also figuring out the NST representation of $\pi$. (By this, I mean the ambinumber $(\pi,0)$, because $\pi$ has no 2-adic meaning.)

When we shift the NST integers $2\lfloor(-3/2)^n\pi/2\rfloor$ and $2\lceil(-3/2)^n\pi/2\rceil$ to the right by $n$ positions, we get the best $n$-place approximations to $(\pi,0)$. This process converges rapidly to the leading digits of the desired value,

$$(\pi,0)=\texttt{21122 21002 20210 00100 02012 02121 12122 02102 00111 01022 01010 10112 12010 22212 11012}\;\ldots\,. \tag{16.1}$$

We can also compute the NST representation of $(0,(\pi/8)^R)$, as well as the leftmost and rightmost paths of the tree $\mathcal{T}_{(\pi/8)^R}$.

But this note is already too long; I'll leave those things as instructive exercises for the reader.

**17. Other things to explore.** The widths $w^{(2n)}$ for integer $n$ are distinct, and it might be interesting to see what happens when they are sorted into increasing order. We noted in (12.3) that $w^{(6n+2)}=\frac{3}{2}w^{(-4n-2)}$; and it follows from (11.3) that

$$w^{(-2)}=\tfrac{1}{2}w^{(0)}. \tag{17.1}$$

Thus $w^{(-4)}$, $w^{(-2)}$, $w^{(0)}$, and $w^{(2)}$ are all expressible in terms of the constant $\lambda$ in (7.4). Are there any interesting relations between $w^{(4)}$, $w^{(8)}$, $w^{(12)}$, $w^{(16)}$, $\ldots$?

One significance of these widths is the fact that the "integer part" of the NST representation of $(x,0)$, when $x$ is a real number, is $2n$ when

$$\lambda^{(2n)}-2n\le x\le\rho^{(2n)}-2n. \tag{17.2}$$

Every even 2-adic number $q$ has a *dual* $q'$ such that $\mathcal{T}_q^R=\mathcal{T}_{q'}$. For example, I found the pairs $(q,q')=(0,-\frac{4}{5})$, $(\frac{8}{5},-\frac{12}{5})$, as well as pairs such as $(-2,\frac{6}{5})$ that are derivable from them. Are other examples of duality expressible in a simple way?

Algorithms (3.3) and (3.4) for negation of finite-precision NST numbers can be extended to infinite precision in a interesting way, operating on "streams" of digits and buffering partial outputs when the final digits aren't yet known. (For example, given two streams of decimal digits that represent real numbers in the normal way, we can add them from left to right if we maintain a "nines counter" when we don't know whether a future carry will occur.)

It would be nice if all ambinumbers could be represented in terms of "balanced" digits '$\bar{1}$', '0', and '1' instead of '0', '1', and '2'. But that doesn't work; for example, the ambinumber $(4,4)$ can't be represented with such a system.

Are there other good ways to represent ambinumbers, instead of using radix $-\frac{3}{2}$ or pairs of numbers with radixes 2 and $\frac{1}{2}$? I tried *negasemiquinary* representation (radix $-\frac{5}{2}$), and found easily that all dyadic rationals have a unique representation with digits $\{0,1,2,3,4\}$. The same is true with the balanced digits $\{-2,-1,0,1,2\}$, in which case we can of course work more simply with radix $+\frac{5}{2}$. But those systems apparently still have a lot of redundancy; they "over-represent" the ambinumbers, and I'm not sure exactly what algebraic system is actually being implemented when we perform arithmetic with such representations.

It might be interesting to explore the system with radix $-2$ and the four *ambinumbers* $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1$ as digits. (We could call this the "nega-ambinary number system.") Or perhaps an ambinumber could be used as the radix.

**18. Conclusion.** Welcome to ambinumber land! In this note I've tried to tell the story of how a newcomer to this territory might begin to explore its main features, using completely elementary methods with the help of a computer. We've seen many ties to other parts of mathematics and computer science.

I fondly hope that the reader now understands why the ambience of ambinumbers has made them so appealing to a connoisseur of trees and arithmetic like myself. This mathematical system surely contains many more secrets that are yet to be revealed.

**Postscript.** See [2] for significant explorations of radix $+3/2$.

After writing the above, I learned from David Eppstein that the set $\mathbb{R} \times \mathbb{Q}_2$ of ambinumbers has been studied before, for example in Section 6.2.1 of [4]. I'll be glad to learn about any other early appearances of this system in the literature.

If I had time — but I definitely *don't* — it would be fun to write a sequel to *Surreal Numbers*, in which Alice and Bill are 81 years old. They're retired; their children live elsewhere; their grandchildren are finishing college and about to get married. They decide to spend their 64th wedding anniversary alone together in a remote part of Svalbard, exploring the ambinumbers. . . .

**References.**


[1] Prasad Chaugule, Nutan Limaye, and Aditya Varre, "Variants of homomorphism polynomials complete for algebraic complexity classes," in *Computing and Combinatorics: 25th International Conference, COCOON 2019*, *Lecture Notes in Computer Science* **11653** (2019), 90–102.

[2] Ben Chen, Richard Chen, Joshua Guo, Tanya Khovanova, Shane Lee, Neil Malur, Nastia Polina, Poonam Sahoo, Anuj Sakarda, Nathan Sheffield, and Armaan Tipimeni, "On base 3/2 and its sequences," arXiv:1808.04304 [math.NT] (13 August 2018), 21 pages. (This preprint is one of the successes of MIT's PRIMES: STEP program for middleschoolers!)

[3] Donald E. Knuth, *Seminumerical Algorithms*. (Reading, Mass.: Addison–Wesley, 1969). (Exercise 4.1–31 has remained essentially unchanged in the third edition (1997).)

[4] Freddie Manners, "A solution to the pyjama problem," *Inventiones Mathematicæ* **202** (2015), 239–270.

[5] Andrew M. Odlyzko and Herbert S. Wilf, "Functional iteration and the Josephus problem," *Glasgow Mathematical Journal* **33** (1991), 235–240.

[6] `http://oeis.org/`, The On-Line Encyclopedia of Integer Sequences®, founded in 1964 by N. J. A. Sloane. Several tables in the present note became sequences A355904–A355913 on 19 September 2022.

[7] Lucía Rossi and Jörg M. Thuswaldner, "A number system with base $-3/2$," *American Mathematical Monthly* **129**, 6 (June–July 2022), 539–553.