Coups de grâceful graphs

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A "KP graph" $K_n \,\square\, P_r$ is a row of $r > 1$ cliques of size $n > 2$. It has rn vertices and $m = r\binom{n}{2} + (r-1)n$ edges. Such a graph is *graceful* if and only if there's an $n \times r$ matrix of distinct integers x_{ij} in the range $0 \le x_{ij} \le m$ for which the m differences

$$\bigl\{|x_{ij} - x_{kj}| \mid 1 \le i < k \le n,\ 1 \le j \le r\bigr\} \cup \bigl\{|x_{ij} - x_{i(j-1)}| \mid 1 \le i \le n,\ 1 < j \le r\bigr\}$$

are themselves distinct. (Since those differences are obviously positive and at most m, they must in fact be a permutation of $\{1, \ldots, m\}$.) For example, $K_5 \,\square\, P_7$ is graceful because of the matrix

$$\begin{pmatrix} 10 & 56 & 99 & 0 & 100 & 13 & 93 \\ 33 & 66 & 7 & 77 & 12 & 87 & 59 \\ 81 & 95 & 1 & 41 & 3 & 94 & 8 \\ 86 & 2 & 97 & 15 & 70 & 26 & 71 \\ 89 & 6 & 79 & 52 & 69 & 45 & 24 \end{pmatrix} : \qquad (*)$$

The differences in its seven columns are

$$\begin{matrix} 23 & 10 & 92 & 77 & 88 & 74 & 34 \\ 71 & 39 & 98 & 41 & 97 & 81 & 85 \\ 76 & 54 & 2 & 15 & 30 & 13 & 22 \\ 79 & 50 & 20 & 52 & 31 & 32 & 69 \\ 48 & 29 & 6 & 36 & 9 & 7 & 51 \\ 53 & 64 & 90 & 62 & 58 & 61 & 12 \\ 56 & 60 & 72 & 25 & 57 & 42 & 35 \\ 5 & 93 & 96 & 26 & 67 & 68 & 63 \\ 8 & 89 & 78 & 11 & 66 & 49 & 16 \\ 3 & 4 & 18 & 37 & 1 & 19 & 47 \end{matrix} ;$$

the differences in its five rows are

$$\begin{matrix} 46 & 43 & 99 & 100 & 87 & 80 \\ 33 & 59 & 70 & 65 & 75 & 28 \\ 14 & 94 & 40 & 38 & 91 & 86 \\ 84 & 95 & 82 & 55 & 44 & 45 \\ 83 & 73 & 27 & 17 & 24 & 21 \end{matrix} ;$$

and those 100 differences do indeed hit each of $\{1, \ldots, 100\}$ exactly once.

Is that amazing, or what?

The gracefulness of KP graphs was first investigated by Lustig and Puget in 2001 [1]; they showed that $K_4 \,\square\, P_2$ is graceful, because of the matrix

$$\begin{pmatrix} 0 & 6 \\ 1 & 15 \\ 5 & 13 \\ 16 & 3 \end{pmatrix} .$$

That work inspired Petrie and Smith shortly afterwards [2] to study $K_n \,\square\, P_2$ for other values of n, because the many symmetries of KP graphs make this problem a good testbed for methods of exploiting symmetry in constraint satisfaction problems.

Indeed, the graceful labeling problem for KP graphs has $4n!$ symmetries, because we can permute the rows of the matrix arbitrarily, and/or reflect it left $\leftrightarrow$ right, and/or replace each entry x_{ij} by $m - x_{ij}$.

When such solutions are considered equivalent, Petrie and Smith showed that $K_3 \,\square\, P_2$ has 4 essentially distinct labelings; that $K_4 \,\square\, P_2$ has 15; and that the following labeling of $K_5 \,\square\, P_2$ is unique:

$$\begin{pmatrix} 0 & 23 \\ 4 & 14 \\ 18 & 6 \\ 19 & 3 \\ 25 & 1 \end{pmatrix}.$$

They discussed several approaches to proving the uniqueness of this remarkable pattern, of which their best took 629 seconds of computer time. This was considered to be a "success story for constraint programming" in those days.

Then in 2010, Smith and Puget [3] devised a much better way to tackle such problems by machine. They needed only 0.2 seconds to reproduce all of the results of Petrie and Smith; and they proved $K_n \,\square\, P_2$ to be ungraceful for $n = (6, 7, 8, 9, 10, 11, 12)$ with running times of only (1, 4, 16, 51, 148, 373, 857) seconds, respectively.

Moreover, Smith and Puget went up to P_3 and even P_4: Their algorithm was able to show that $K_3 \,\square\, P_3$, $K_4 \,\square\, P_3$, and $K_5 \,\square\, P_3$ have respectively 284, 704, and 101 essentially distinct solutions, while $K_3 \,\square\, P_4$ has 12754. Furthermore, they discovered — after 17000 seconds of computation — that $K_6 \,\square\, P_3$ is *uniquely* graceful!

I learned of their stunning results a year ago, and of course I was keen to see a matrix for that unique solution. Curiously, they had chosen not to include it in their paper, presumably because they were interested more in methods than in answers. (An admirable trait; but such an answer deserves to be broadcast.)

As usual, I had lots of other things to do at the time. (Who doesn't?) The prospect of implementing their somewhat complex method and then waiting about 4.7 hours for an answer was not very appealing. So I wrote to Smith and Puget, asking if they had saved any notes from their work a decade earlier.

The answer turned out to be "No." Thus, in February 2020, I could no longer resist the challenge of finding that elusive labeling of $K_6 \,\square\, P_3$. And I got lucky: After studying their methods closely, I realized that a new method, different from anything I'd seen before, would actually work much better. In fact, my program BACK-GRACEFUL-KMP3 [4] was able to find the 101 solutions for $K_5 \,\square\, P_3$ in only 12 seconds, compared to their 1020. And it needed only 174 seconds to find the unique matrix that I sought:

$$\begin{pmatrix} 0 & 56 & 1 \\ 5 & 36 & 9 \\ 12 & 6 & 52 \\ 33 & 55 & 26 \\ 44 & 2 & 49 \\ 57 & 20 & 11 \end{pmatrix}.$$

Moreover, six more runs of the algorithm established the ungracefulness of $K_n \,\square\, P_3$ for $7 \le n \le 12$, needing at most 1.3 hours per problem.

A similar but much simpler program called BACK-GRACEFUL-KMP2 [5] was in fact able to establish the unique labeling for $K_5 \,\square\, P_2$ in only 0.004 seconds! It also verified ungracefulness of $K_6 \,\square\, P_2, \ldots, K_{12} \,\square\, P_2$ in less than 0.012 seconds per problem.

Additional light now began to dawn, because it wasn't hard to see that the computer was performing essentially the same computations over and over, for all sufficiently large values of n. With small changes to my programs, the new algorithm was therefore able to prove that $K_n \,\square\, P_2$ is ungraceful for *all* values of $n > 5$; moreover, $K_n \,\square\, P_3$ is ungraceful for all values of $n > 6$. The former result is based on a tree of size 8910, found in 0.12 seconds; the latter is based on a tree of size 5,463,149,994, found in 4711 seconds.

Smith and Puget went on to consider KC graphs, namely $K_n \square C_r$ for $n > 2$ and $r > 2$. Their main result, obtained in 2845 seconds, was that $K_6 \,\square\, C_3$ isn't graceful. Straightforward modifications of my programs (see [7]) establish this in 25 seconds, and prove the ungracefulness of $K_n \,\square\, C_3$ for all $n > 5$.

Of course that's not the end of the story. Indeed, the 5×7 matrix in $(*)$ above is quite different from any examples of $K_n \square P_r$ labeling that have appeared previously in the literature. And a completely different method was used to discover it.

Last month Tomas Rokicki and I developed a Las Vegas algorithm that seems to be unusually effective for graceful labeling problems on graphs with at most (say) 50 vertices and about 100 edges. We developed it for unsymmetric graphs, but it apparently gives good results also when there's lots of symmetry. The program is online [8] and available for experimentation. Tomas used it to discover (∗) on 14 November 2020. It has recently established the gracefulness of several other KP graphs and KC graphs whose status was previously unknown.

Here is a chart that shows what I currently know about KP graphs, gracefulnesswise, based on these new results. (I leave out cases with $r \le 3$, since they have now been completely analyzed.)

	$r=4$	$r=5$	$r=6$	$r=7$	$r=8$	$r=9$	$r=10$	$r=11$	$r=12$
$n=3$	12754	E	E	E	E	E	E	E	E
$n=4$	164273	SP	SP	SP	SP	KR	KR	KR	KR
$n=5$	KR	KR	KR	KR	?	.	.	.	.
$n=6$	KR	?	.	.	.	.	.	.	.
$n=7$	?	.	.	.	.	.	.	.	.

A numerical entry means this is the exact number of graceful labelings. (The value 12754 is from [3] and I've confirmed it. I came up with the value 164273 yesterday, and it's not yet confirmed by anybody else.) 'E' means that graceful labelings are easy to find; indeed, program [8] spits them out faster than it can write them to a file. (I believe there are exponentially many as $r \to \infty$.) 'SP' means that a solution was reported in [3]. 'KR' means that a solution was found by [8]. '?' means that I've tried [8] for some hours without success; Tomas may still be running [8] in the background on his computers, hoping to find a lucky solution. '.' means that I've never dared to try this case yet. 'FS' means found by Filip Stappers (see the appendix).

And here's a similar chart for KC gracefulness.

	$r=4$	$r=5$	$r=6$	$r=7$	$r=8$	$r=9$	$r=10$	$r=11$	$r=12$
$n=3$	3809	0	E	0	E	0	E	0	E
$n=4$	SP	SP	SP	KR	KR	KR	KR	KR	FS
$n=5$	SP	KR	0	0	.	.	0	0	.
$n=6$	KR	?	.	.	.	.	.	.	.
$n=7$	?	.	.	.	.	.	.	.	.

The 0's follow from a well-known parity condition: The graph $K_n \,\square\, C_r$ is regular of degree $n+1$, hence its vertices all have even degree when n is odd. In such cases the graph cannot be graceful if the number of edges, m, has the form $4k+1$ or $4k+2$. This rules out odd r when $n=3$; it rules out $r \bmod 4 \in \{2,3\}$ when $n=5$; there's no parity condition when $n=7$. (Smith and Puget erroneously stated that n and r cannot both be odd; but $K_5 \,\square\, C_5$ *is* graceful.)

I'm going to explain [4] and [5] and [6] and [7] in Section 7.2.2.3 of *The Art of Computer Programming*, some day; a (very) preliminary draft of the relevant portion is available for proofreading [9]. But there won't be room in that book to give full details of all the graceful cases. Therefore I shall conclude this note by presenting matrices that prove gracefulness in all of the cases marked SP or KR in the charts above.

First, $K_4 \,\square\, P_r$:

$$\begin{pmatrix} 29 & 3 & 34 & 1 \\ 12 & 31 & 7 & 0 \\ 4 & 33 & 11 & 21 \\ 18 & 15 & 2 & 36 \end{pmatrix} \quad \begin{pmatrix} 11 & 24 & 16 & 46 & 29 \\ 37 & 6 & 39 & 3 & 40 \\ 32 & 12 & 41 & 0 & 44 \\ 18 & 34 & 7 & 45 & 5 \end{pmatrix} \quad \begin{pmatrix} 48 & 10 & 58 & 22 & 64 & 1 & 4 \\ 14 & 49 & 31 & 9 & 52 & 15 & 55 \\ 47 & 26 & 51 & 7 & 35 & 65 & 0 \\ 38 & 57 & 5 & 63 & 3 & 60 & 66 \end{pmatrix} \quad \begin{pmatrix} 68 & 34 & 62 & 35 & 73 & 13 & 1 & 15 \\ 22 & 69 & 39 & 71 & 18 & 57 & 6 & 67 \\ 42 & 58 & 29 & 54 & 9 & 75 & 3 & 74 \\ 64 & 21 & 70 & 14 & 72 & 7 & 76 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 65 & 1 & 70 & 34 & 17 & 62 & 13 & 46 \\ 0 & 76 & 2 & 68 & 37 & 16 & 29 & 47 \\ 71 & 4 & 58 & 6 & 67 & 9 & 53 & 24 \\ 14 & 74 & 11 & 49 & 12 & 51 & 61 & 20 \end{pmatrix} \quad \begin{pmatrix} 20 & 68 & 31 & 76 & 11 & 38 & 82 & 0 & 85 \\ 66 & 28 & 70 & 48 & 14 & 73 & 13 & 86 & 15 \\ 49 & 17 & 74 & 12 & 61 & 79 & 3 & 81 & 1 \\ 40 & 47 & 16 & 24 & 77 & 5 & 80 & 25 & 2 \end{pmatrix} \quad \begin{pmatrix} 91 & 9 & 87 & 7 & 77 & 50 & 18 & 72 & 51 & 69 \\ 1 & 3 & 13 & 65 & 89 & 14 & 62 & 25 & 81 & 58 \\ 95 & 28 & 73 & 12 & 55 & 17 & 82 & 33 & 68 & 27 \\ 0 & 96 & 4 & 93 & 5 & 90 & 11 & 88 & 22 & 84 \end{pmatrix}$$

$$\begin{pmatrix} 85 & 14 & 71 & 83 & 0 & 105 & 19 & 41 & 88 & 15 & 89 \\ 61 & 46 & 100 & 11 & 106 & 99 & 33 & 96 & 12 & 90 & 28 \\ 18 & 82 & 23 & 103 & 3 & 94 & 54 & 87 & 35 & 65 & 20 \\ 44 & 95 & 13 & 101 & 102 & 1 & 98 & 2 & 72 & 34 & 62 \end{pmatrix} \quad \begin{pmatrix} 74 & 47 & 75 & 25 & 107 & 31 & 6 & 112 & 8 & 56 & 93 & 62 \\ 30 & 87 & 36 & 85 & 11 & 109 & 24 & 116 & 2 & 97 & 4 & 92 \\ 83 & 21 & 55 & 17 & 72 & 7 & 115 & 0 & 113 & 10 & 90 & 20 \\ 51 & 94 & 19 & 96 & 13 & 76 & 105 & 15 & 3 & 110 & 26 & 78 \end{pmatrix}$$

Next, $K_5 \mathbin{\square} P_r$ and $K_6 \mathbin{\square} P_r$ (see also (∗) above):

$$\begin{pmatrix} 30 & 52 & 7 & 31 \\ 4 & 37 & 8 & 20 \\ 23 & 55 & 16 & 47 \\ 53 & 2 & 50 & 6 \\ 40 & 0 & 54 & 26 \end{pmatrix} \quad \begin{pmatrix} 63 & 5 & 64 & 8 & 70 \\ 33 & 13 & 41 & 24 & 2 \\ 6 & 59 & 9 & 50 & 3 \\ 19 & 44 & 4 & 69 & 0 \\ 57 & 23 & 52 & 17 & 66 \end{pmatrix} \quad \begin{pmatrix} 43 & 16 & 56 & 24 & 1 & 58 \\ 48 & 77 & 22 & 82 & 8 & 80 \\ 62 & 25 & 81 & 3 & 84 & 15 \\ 78 & 31 & 73 & 0 & 85 & 5 \\ 12 & 79 & 11 & 44 & 72 & 54 \end{pmatrix} \quad \begin{pmatrix} 25 & 61 & 9 & 38 \\ 33 & 71 & 27 & 1 \\ 72 & 18 & 76 & 5 \\ 8 & 6 & 75 & 16 \\ 53 & 11 & 41 & 62 \\ 2 & 74 & 0 & 78 \end{pmatrix}$$

Next, $K_4 \mathbin{\square} C_r$:

$$\begin{pmatrix} 8 & 33 & 2 & 35 \\ 4 & 17 & 0 & 36 \\ 15 & 12 & 22 & 7 \\ 27 & 3 & 40 & 1 \end{pmatrix} \quad \begin{pmatrix} 6 & 50 & 0 & 47 & 11 \\ 48 & 9 & 46 & 14 & 21 \\ 26 & 49 & 31 & 2 & 32 \\ 10 & 1 & 3 & 28 & 45 \end{pmatrix} \quad \begin{pmatrix} 2 & 57 & 48 & 26 & 1 & 59 \\ 49 & 19 & 7 & 12 & 60 & 20 \\ 23 & 42 & 5 & 54 & 0 & 3 \\ 33 & 6 & 40 & 8 & 53 & 9 \end{pmatrix} \quad \begin{pmatrix} 27 & 64 & 2 & 66 & 1 & 69 & 42 \\ 57 & 6 & 41 & 0 & 70 & 9 & 8 \\ 7 & 61 & 25 & 53 & 60 & 13 & 51 \\ 38 & 32 & 65 & 48 & 3 & 55 & 30 \end{pmatrix}$$

$$\begin{pmatrix} 27 & 63 & 9 & 22 & 45 & 5 & 75 & 38 \\ 1 & 72 & 21 & 50 & 2 & 78 & 0 & 80 \\ 61 & 17 & 56 & 18 & 76 & 73 & 57 & 16 \\ 60 & 7 & 70 & 43 & 26 & 11 & 77 & 8 \end{pmatrix} \quad \begin{pmatrix} 26 & 78 & 90 & 2 & 83 & 25 & 51 & 18 & 61 \\ 86 & 3 & 87 & 8 & 79 & 30 & 74 & 58 & 13 \\ 9 & 85 & 0 & 80 & 15 & 81 & 20 & 73 & 34 \\ 39 & 11 & 1 & 71 & 33 & 62 & 40 & 16 & 75 \end{pmatrix}$$

$$\begin{pmatrix} 14 & 35 & 89 & 34 & 71 & 41 & 88 & 55 & 99 & 92 \\ 96 & 13 & 93 & 20 & 73 & 12 & 62 & 28 & 79 & 0 \\ 1 & 94 & 4 & 76 & 7 & 64 & 26 & 86 & 2 & 100 \\ 77 & 3 & 44 & 9 & 25 & 40 & 23 & 98 & 11 & 6 \end{pmatrix} \quad \begin{pmatrix} 1 & 110 & 20 & 89 & 14 & 97 & 29 & 77 & 94 & 53 & 102 \\ 98 & 108 & 43 & 104 & 42 & 16 & 96 & 23 & 54 & 19 & 46 \\ 5 & 0 & 6 & 26 & 72 & 59 & 84 & 24 & 75 & 11 & 93 \\ 105 & 3 & 109 & 10 & 101 & 27 & 51 & 95 & 18 & 103 & 7 \end{pmatrix}$$

And finally $K_5 \mathbin{\square} C_r$, $K_6 \mathbin{\square} C_r$:

$$\begin{pmatrix} 57 & 13 & 47 & 14 \\ 0 & 59 & 10 & 41 \\ 58 & 23 & 55 & 3 \\ 4 & 52 & 5 & 26 \\ 60 & 43 & 29 & 54 \end{pmatrix} \quad \begin{pmatrix} 74 & 2 & 71 & 14 & 8 \\ 3 & 18 & 69 & 10 & 33 \\ 41 & 70 & 23 & 59 & 20 \\ 73 & 9 & 43 & 24 & 51 \\ 0 & 62 & 6 & 64 & 75 \end{pmatrix} \quad \begin{pmatrix} 77 & 2 & 74 & 4 \\ 58 & 8 & 61 & 9 \\ 0 & 63 & 23 & 68 \\ 81 & 39 & 53 & 21 \\ 1 & 30 & 5 & 83 \\ 84 & 73 & 7 & 48 \end{pmatrix}$$

[1] Irvin J. Lustig and Jean-François Puget, "Program does not equal program: Constraint programming and its relationship to mathematical programming," *INFORMS Journal on Applied Analytics* **31**, 6 (November/December 2001), 29–53 [especially pages 42–43].

[2] Karen E. Petrie and Barbara M. Smith, "Symmetry breaking in graceful graphs," *Lecture Notes in Computer Science* **2833** (2003), 930–934.

[3] Barbara M. Smith and Jean-François Puget, "Constraint models for graceful graphs," *Constraints* **15** (2010), 64–92 [especially pages 78–82].

[4] `http://cs.stanford.edu/~knuth/programs/back-graceful-kmp3.w`
[posted 16 Nov 2020].

[5] `http://cs.stanford.edu/~knuth/programs/back-graceful-kmp2.w`
[posted 16 Nov 2020].

[6] `http://cs.stanford.edu/~knuth/programs/back-graceful-kmp2-largem.ch`
and `http://cs.stanford.edu/~knuth/programs/back-graceful-kmp3-largem.ch`
[posted 30 Aug 2020].

[7] `http://cs.stanford.edu/~knuth/programs/back-graceful-kmc3.ch`
and `http://cs.stanford.edu/~knuth/programs/back-graceful-kmc3-largem.ch`
[posted 13 Nov 2020].

[8] `http://cs.stanford.edu/~knuth/programs/back-graceful-rooted-randomrestarts.w`
[posted 12 Oct 2020].

[9] Donald E. Knuth, `http://cs.stanford.edu/~knuth/fasc7a.ps.gz`
[posted 25 Oct 2020 and updated periodically].

Appendix (late-breaking news)!

Here's $K_4 \,\square\, C_{12}$, found by Filip Stappers on 24 November 2020:

$$\begin{pmatrix} 91 & 65 & 22 & 61 & 38 & 96 & 103 & 5 & 117 & 12 & 115 & 19 \\ 64 & 17 & 98 & 16 & 51 & 85 & 48 & 67 & 1 & 114 & 14 & 92 \\ 31 & 87 & 47 & 99 & 9 & 102 & 18 & 113 & 2 & 109 & 35 & 110 \\ 23 & 33 & 83 & 30 & 95 & 8 & 107 & 3 & 120 & 0 & 106 & 43 \end{pmatrix}$$