Parades and Poly-Bernoulli Bijections

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This is a story about a beautiful array of numbers that arises in an astonishing number of interesting combinatorial contexts. It's a counterexample to the hypothesis that all of the important "special numbers" were discovered long, long ago — because the earliest known appearance of this particular array was in 1997. It has, however, been rediscovered several times since then.

The numbers in question, which we shall call $B_{m,n}$ in this note, begin as follows:

$B_{m,n}$	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	
$m=0$	1	1	1	1	1	1	1	1	
$m=1$	1	2	4	8	16	32	64	128	
$m=2$	1	4	14	46	146	454	1394	4246	
$m=3$	1	8	46	230	1066	4716	20266	85310	
$m=4$	1	16	146	1066	6902	41506	237686	1315666	(0.1)
$m=5$	1	32	454	4718	41506	329462	2441314	17234438	
$m=6$	1	64	1394	20266	237686	2441314	22934774	202229266	
$m=7$	1	128	4246	85310	1315666	17234438	202229266	2193664790	

Notice that we have diagonal symmetry, $B_{m,n} = B_{n,m}$, throughout this table.

The sequences for $m = 0$ and $m = 1$ are familiar. When $m = 2$ the sequence isn't so well known, although it turns out that Euler mentioned those numbers in Section 216 of the calculus book [10] that he published in 1748. Then when $m = 3$ the array begins to break new ground; the historic number 1066 must be there just by coincidence.

In this note we'll see that $B_{m,n}$ is the number of combinatorial configurations of many different kinds. For example, it's the number of ways to assign directions to the edges of the complete bipartite graph $K_{m,n}$, in such a way that no oriented cycles arise. It's also the number of permutations $p_1p_2\ldots p_{m+n}$ of $\{1,2,\ldots,m+n\}$ for which we have $k-m \le p_k \le k+n$, for all k. It's the number of binary relations $\smile$ between a variable $x \in \{1,\ldots,m\}$ and a variable $y \in \{1,\ldots,n\}$ such that $x \smile y$ and $x' \smile y'$ implies $\max(x,x') \smile \max(y,y')$. And so on(!).

Furthermore, we'll see that there are relatively simple *bijections* (one-to-one correspondences) between the objects of each kind: Acyclic orientations correspond to classes of permutations, which correspond to classes of relations, etc. The techniques of contriving such bijections are in fact interesting in themselves.

A new kind of pattern, which we shall call a "parade" of m girls and n boys, turns out to be a particularly insightful way to understand the combinatorial configurations that are enumerated by the numbers $B_{m,n}$.

A few exercises have been included for self-study, with answers at the end.

1. Definitions. Our story begins with Masanobu Kaneko's paper [1], which introduced a nice generalization of the classic Bernoulli numbers: Let $B_n^{(s)}$ be the sequence of coefficients defined by the formal infinite series

$$\sum_{n\ge 0} B_n^{(s)} \frac{z^n}{n!} \;=\; \frac{1}{1-e^{-z}} \sum_{k\ge 1} \frac{(1-e^{-z})^k}{k^s}, \tag{1.1}$$

where s is any complex number. Kaneko called $B_n^{(s)}$ a *poly-Bernoulli number*, because $B_n^{(1)}$ is the famous sequence published by Jakob Bernoulli in 1713 (with $B_1^{(1)} = +1/2$), and because $\sum_{k\ge1} z^k/k^s$ is called the polylogarithm function $\mathrm{Li}_s(z)$. Indeed, when we set $s = 1$ in (1.1) we get the ordinary logarithm, and the right-hand side simplifies to

$$\frac{\ln\bigl(1/(1-(1-e^{-z}))\bigr)}{1-e^{-z}} \;=\; \frac{z}{1-e^{-z}} \;=\; 1 + \frac{z}{2} + \frac{z^2}{12} - \frac{z^4}{720} + \cdots, \tag{1.2}$$

which is the exponential generating function that defines *non*-poly Bernoulli numbers. The name 'poly-Bernoulli' is a bit of a jawbreaker; let's refer to $B_n^{(s)}$ as a 'pB number', for short.

Kaneko was motivated purely by considerations of abstract number theory, without any hint of applications to combinatorics. The main question that he put to himself at the time was to determine the prime factorization of the denominator of $B_n^{(2)}$, because he knew that the analogous question for $B_n^{(1)}$ has a very interesting answer. (See, for example, exercise 6.54 in [5].)

Students of generatingfunctionology know that one of the most important formulas is

$$\frac{(e^z-1)^m}{m!} \;=\; \sum_{n\ge0} \left\{ {n \atop m} \right\} \frac{z^n}{n!}\,, \tag{1.3}$$

the exponential generating function for Stirling partition numbers $\left\{ {n \atop m} \right\}$ when m is fixed. (See, for instance, Eq. (7.49) in [5].) Plugging this into (1.1) yields

$$\sum_{n\ge0} B_n^{(s)} \frac{z^n}{n!} \;=\; \sum_{k\ge0} \frac{(1-e^{-z})^k}{(k+1)^s} \;=\; \sum_{k\ge0} k!\frac{(-1)^k}{(k+1)^s} \sum_{n\ge0} \left\{ {n \atop k} \right\} \frac{(-z)^n}{n!}\,;$$

so we have an explicit way to express every pB number as a sum:

$$B_n^{(s)} \;=\; \sum_{k\ge0} k!\left\{ {n \atop k} \right\} \frac{(-1)^{n+k}}{(k+1)^s}. \tag{1.4}$$

One consequence of this formula is that $B_n^{(s)}$ turns out to be an integer whenever s is a negative integer. (On the other hand, $B_1^{(s)} = 2^{-s}$ is noninteger whenever $s>0$.) In fact, the array (0.1) is obtained in this way when $s=-m$: We define

$$B_{m,n} \;=\; B_n^{(-m)} = \sum_{k\ge0} (-1)^{n+k} k! \left\{ {n \atop k} \right\} (k+1)^m\,. \tag{1.5}$$

For example, the special case

$$B_{m,2} \;=\; -\left\{ {2 \atop 1} \right\} 2^m + 2\left\{ {2 \atop 2} \right\} 3^m \;=\; 2\cdot 3^m - 2^m \tag{1.6}$$

gives the numbers 1, 4, 14, 46, ... that appear in the column for $n=2$ in (0.1).

But wait a minute. Formula (1.5) sure doesn't look symmetrical in m and n; yet we know from Table (0.1) that $B_{n,m}$ is actually *equal* to $B_{m,n}$, at least when m and n are small. Such all-pervasive symmetry simply cannot be a fluke! And indeed, there's an elegant way to verify that symmetry does hold, for all m and for all n, by examining a *bivariate* generating function, $G(w,z) = \sum_{m,n\ge0} B_{m,n} \frac{w^m}{m!}\frac{z^n}{n!}$, namely the generating function for $B_{m,n}$ that is simultaneously exponential in *both* parameters.

$$\begin{aligned} G(w,z) = \sum_{m\ge0}\frac{w^m}{m!}\sum_{n\ge0} B_{m,n}\frac{z^n}{n!} &= \sum_{m\ge0}\frac{w^m}{m!}\sum_{k\ge0}(1-e^{-z})^k(k+1)^m = \sum_{k\ge0}(1-e^{-z})^k e^{(k+1)w} \\ &= e^w\sum_{k\ge0}(e^w-e^{w-z})^k = \frac{e^w}{1-(e^w-e^{w-z})} = \frac{e^w}{e^{w-z}+1-e^w}\,. \end{aligned}$$

We have proved that the bivariate generating function is nicely symmetric in w and z:

$$G(w,z) \;=\; \frac{e^{w+z}}{e^w+e^z-e^{w+z}}\,. \tag{1.7}$$

Things are looking up, because we can now deduce a symmetrical way to compute $B_{m,n}$, in place of the unsymmetrical (1.5). We can write

$$G(w,z) \;=\; \frac{e^{w+z}}{1-(e^w-1)(e^z-1)} = \sum_{k\ge0} e^w(e^w-1)^k(e^z-1)^k e^z; \tag{1.8}$$

so we'd like to understand the generating function $e^w(e^w-1)^k$. (See [21].) The derivative of (1.3) with respect to z reveals the answer to that question:

$$\frac{e^z(e^z-1)^m}{m!} \;=\; \sum_{n\ge0}\left\{ {n+1 \atop m+1} \right\}\frac{z^n}{n!}\,. \tag{1.9}$$

Consequently we have

$$\sum_{m,n\ge0} B_{m,n}\frac{w^m}{m!}\frac{z^n}{n!} \;=\; G(w,z) \;=\; \sum_{k\ge0}\Bigl(k!\sum_{m\ge0}\left\{{m+1\atop k+1}\right\}\frac{w^m}{m!}\Bigr)\Bigl(k!\sum_{n\ge0}\left\{{n+1\atop k+1}\right\}\frac{z^n}{n!}\Bigr)\,. \tag{1.10}$$

And by equating the coefficients of $w^m z^n$ on each side, we obtain the symmetric formula that we seek,

$$B_{m,n} \;=\; \sum_{k\ge0} k!^2\left\{{m+1\atop k+1}\right\}\left\{{n+1\atop k+1}\right\}. \tag{1.11}$$

For example, $B_{3,4}=0!^2\left\{{4\atop 1}\right\}\left\{{5\atop 1}\right\}+1!^2\left\{{4\atop 2}\right\}\left\{{5\atop 2}\right\}+2!^2\left\{{4\atop 3}\right\}\left\{{5\atop 3}\right\}+3!^2\left\{{4\atop 4}\right\}\left\{{5\atop 4}\right\}=1\cdot1\cdot1+1\cdot7\cdot15+4\cdot6\cdot25+36\cdot1\cdot10=1066$.

2. A combinatorial interpretation. So far we've been happily manipulating algebraic formulas, without regard to what those formulas might actually mean. But the Stirling partition numbers $\left\{{n\atop m}\right\}$ do have a simple meaning: They're the number of ways to partition a set of n elements into m disjoint nonempty subsets.

Of course the coefficients of $z^n/n!$ in the exponential generating function e^z-1 have an even simpler meaning: They're the number of ways to give n labels to a nonempty set with n elements, namely $n!$ when $n>0$ but 0 when $n=0$.

The "symbolic method" of combinatorial enumeration now explains why (1.3) is true: When n elements are partitioned into m disjoint nonempty subsets $S_1,\ \ldots,\ S_m$, the number of ways to label the elements of those labeled subsets has the exponential generating function $(e^z-1)^m$. And we divide by $m!$, because all permutations of $\{S_1,\ldots,S_m\}$ give the same set partition. (See, for instance, the elegant exposition of the "symbolic method" in [7], formulas II-(13) and II-(14).)

Similarly, (1.9) tells us that $m!\left\{{n+1\atop m+1}\right\}$ is the number of ways to partition an n-element set into disjoint subsets $\{S_0,S_1,\ldots,S_m\}$, where S_0 might be empty but the other sets $S_1,\ \ldots,\ S_m$ must be nonempty. It's true because we can add an $(n+1)$st "dummy" element, then find all $\left\{{n+1\atop m+1}\right\}$ partitions of the extended set into $m+1$ nonempty blocks, afterwards letting S_0 be the elements that belong to the same block as the dummy element.

(Equivalently, there are $m!\left\{{n+1\atop m+1}\right\}$ ways to choose m disjoint nonempty subsets of an n-element set, without necessarily covering all n elements. The uncovered elements are S_0.)

OK then, what does the nice symmetrical formula (1.11) mean? Its term for k comes from the term $e^w(e^w-1)^k(e^z-1)^k e^z$ in (1.8). Therefore we can interpret it as the number of ways to present an m-element set S as a disjoint union $S=S_0\cup S_1\cup\cdots\cup S_d$ and an n-element set T as a disjoint union $T=T_1\cup\cdots\cup T_d\cup T_{d+1}$, where S_0 and/or T_{d+1} might be empty but the other subsets must be nonempty.

That sounds pretty abstract. Fortunately there's a much more concrete way to describe the same setup, which we shall call a "girls and boys parade." There are m girls $\{g_1,\ldots,g_m\}$ and n boys $\{b_1,\ldots,b_n\}$, where g_i is younger than g_{i+1} and b_j is younger than b_{j+1}, but we know nothing about the relative ages of g_i and b_j. In how many ways can they all line up in a sequence such that no girl is directly preceded by an older girl and no boy is directly preceded by an older boy?

The answer is $B_{m,n}$. For example, here are the $B_{2,2}=14$ possible parades of two girls and two boys:

$$\begin{gathered} g_1g_2b_1b_2,\ g_1b_1g_2b_2,\ g_1b_1b_2g_2,\ g_1b_2g_2b_1,\ g_2b_1g_1b_2,\ g_2b_1b_2g_1,\ g_2b_2g_1b_1,\\ b_1g_1g_2b_2,\ b_1g_1b_2g_2,\ b_1g_2b_2g_1,\ b_1b_2g_1g_2,\ b_2g_1g_2b_1,\ b_2g_1b_1g_2,\ b_2g_2b_1g_1. \end{gathered} \tag{2.1}$$

To see why this works in general, we merely need to notice that every parade can be written in the form $S_0T_1S_1\ldots T_dS_dT_{d+1}$, where $S_0\cup S_1\cup\cdots\cup S_d$ is a disjoint union of the girls and $T_1\cup\cdots\cup T_d\cup T_{d+1}$ is a disjoint union of the boys; S_0 and/or T_{d+1} might be empty, but the other subsets are nonempty; girls and boys within a subset appear from youngest to oldest. The value of d is the number of times a boy is directly followed by a girl, and we say that d is the *order* of the parade. (The respective values of d in (2.1) are 0, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 1, 1, 2, 2.)

In this note we shall let $\mathcal{P}_{m,n}$ be the set of all possible parades that can be formed by m labeled girls and n labeled boys. Of course $\mathcal{P}_{2,2}$ is too small to give a feeling for parades in general; here's a more typical example, taken more or less at random from $\mathcal{P}_{16,20}$:

$$\Pi = b_6 b_{13} g_2 g_4 b_5 b_{16} g_7 b_{15} g_1 g_{10} g_{16} b_{10} g_3 b_2 b_7 b_{14} g_5 g_9 g_{11} b_8 b_9 b_{18} b_{20} g_8 b_3 b_4 g_{14} b_{17} g_{12} b_{11} g_6 g_{13} g_{15} b_1 b_{12} b_{19}\,. \qquad (2.2)$$

It's a parade of order 9.

Inside a computer, there's a nice way to represent a parade as two digit strings $s_0 s_1 \ldots s_m$ and $t_0 t_1 \ldots t_n$: Girl g_i belongs to set S_{s_i} and boy b_j belongs to set T_{t_j}. Thus $s_0 = t_0 = 0$, and the other digits are between 0 and d; every digit from 1 to d occurs at least once. For example, the parade Π in (2.2) is represented by the digit strings

$$s_0 s_1 \ldots s_{16} = 03141592653589793 \qquad \text{and} \qquad t_0 t_1 \ldots t_{20} = 005772156649015328606 \qquad (2.3)$$

(so you might guess that it's not a *truly* random example).

A valid digit string can, in turn, be characterized by a permutation σ of $[1\,..\,d]$ and a *restricted growth string* $\Sigma = a_0 a_1 \ldots a_m$, where a restricted growth string has $a_0 = 0$ and $0 \le a_{i+1} \le \max\{a_0, \ldots, a_i\}$ for $0 \le i < m$. (Restricted growth strings are the method of choice for representing set partitions in programs; see [23, §7.2.1.5].) The corresponding digit string has $s_i = a_{i\sigma}$, where $0\sigma = 0$. For example, the permutations and restricted growth strings that correspond to (2.3) are

$$(\sigma = 314592687,\ \Sigma = 01232456741485951) \quad \text{and} \quad (\tau = 572164938,\ \mathrm{T} = 001223415567041839505). \qquad (2.4)$$

Parades turn out to be quite delightful and instructive combinatorial patterns. The more one studies them, the more one agrees with Harold Arlen, when he wrote "I love a parade" in 1932!

3. Ranking and unranking. The patterns of a finite combinatorial class $\mathcal{A}$ can always be listed in some order: $\alpha_0, \alpha_1, \ldots, \alpha_{|\mathcal{A}|-1}$. This ordering, when described in high-falutin' mathematical jargon, is a *bijection* between $\mathcal{A}$ and the integers $[0\,..\,|\mathcal{A}|) = \{0, 1, \ldots, |\mathcal{A}| - 1\}$. The process of looking at a given pattern α_k and discovering its index k in this correspondence is called *ranking*; the inverse problem, which determines the pattern α_k when its index k is given, is called *unranking*.

For example, there are 2^n binary n-tuples $b_1 \ldots b_n$, where each b_j is either 0 or 1. So there are $2^n!$ bijections between binary n-tuples and the numbers of the half-open interval $[0\,..\,2^n)$. Most of those bijections are pretty weird and unimportant. But one of them is quite natural and useful, namely to let $b_1 \ldots b_n$ correspond to the integer whose representation in the binary number system is $(b_1 \ldots b_n)_2$. In particular, the 4-tuple 1101 has rank $(1101)_2 = 13$; conversely, the 4-tuple α_{13} is 1101. (This bijection corresponds to lexicographic order of the n-tuples. A similar one, where $b_1 \ldots b_n \leftrightarrow (b_n \ldots b_1)_2$, corresponds to "colexicographic order.")

Suppose $\mathcal{A}$ is the disjoint union $\mathcal{A}' \cup \mathcal{A}''$ of classes whose rank functions are rank$'$ and rank$''$. Then $|\mathcal{A}| = |\mathcal{A}'| + |\mathcal{A}''|$, and it's natural to define

$$\mathrm{rank}(\alpha) \;=\; \begin{cases} \mathrm{rank}'(\alpha), & \text{if } \alpha \in \mathcal{A}'; \\ |\mathcal{A}'| + \mathrm{rank}''(\alpha), & \text{if } \alpha \in \mathcal{A}''. \end{cases} \qquad (3.1)$$

Similarly, if $\mathcal{A}$ is representable as a Cartesian product $\mathcal{A}' \times \mathcal{A}''$, we have $|\mathcal{A}| = |\mathcal{A}'|\,|\mathcal{A}''|$, and we can define

$$\mathrm{rank}((\alpha', \alpha'')) \;=\; \mathrm{rank}'(\alpha')|\mathcal{A}''| + \mathrm{rank}''(\alpha''). \qquad (3.2)$$

(This rule corresponds to lexicographic order, and to a mixed-radix number system with radices $|\mathcal{A}'|$ and $|\mathcal{A}''|$.) Unranking is easy in both (3.1) and (3.2).

Our principal goal in this note is to discover useful bijections between many combinatorial classes $\mathcal{A}$ for which $|\mathcal{A}| = B_{m,n}$. In particular, we should be able to find a fairly natural bijection between $\mathcal{P}_{m,n}$ and the integers $[0\,..\,B_{m,n})$.

For example, what is the rank of the "typical" parade Π in (2.2)? To answer this question, we need to know the ranks of the permutations and restricted growth strings σ, Σ, τ, and T in (2.4).

There are $(9!)! \approx 1.6 \times 10^{1859933}$ ways to rank the permutations of $\{1,2,3,4,5,6,7,8,9\}$; we shall choose lexicographic order. Then ranks are readily computed from the "inversion table" $C_1 \ldots C_9$, where $C_j = |\{i \mid i > j \text{ and } p_i < p_j\}|$. (See exercise 5.1.1–7 in [3].) Indeed, the formula

$$\operatorname{rank}(C_1 \ldots C_n) = ((\ldots((C_1(n-1)) + C_2)(n-2) + \cdots) + C_{n-1})1 + C_n \;=\; \sum_{j=1}^{n} C_j (n-j)! \tag{3.3}$$

arises from the recurrence $n! = n(n-1)!$ using (3.2). The inversion tables for σ and τ are respectively 201140010 and 451021200; so their ranks turn out to be 81577 and 187258.

The basic recurrence for Stirling partition numbers,

$$\left\{{m+1 \atop d+1}\right\} \;=\; (d+1)\left\{{m \atop d+1}\right\} + \left\{{m \atop d}\right\}, \tag{3.4}$$

is based on the fact that a partition of m girls into $d+1$ nonempty blocks either puts the oldest girl into a $(d+1)$-block partition of the younger ones or into a new block by herself. It leads via (3.1) and (3.2) to a slick way to compute the rank r of any given restricted growth sequence $a_0 a_1 \ldots a_n$:

$$\begin{aligned}&\text{“Set } r \leftarrow d \leftarrow 0, \text{ and do the following for } j = 1, \ldots, n\text{:}\\ &\text{If } a[j] > d, \text{ set } d \leftarrow d+1 \text{ and } r \leftarrow r + (d+1)\left\{{j \atop d+1}\right\};\\ &\text{otherwise set } r \leftarrow r + a_j \left\{{j \atop d+1}\right\}.\text{”}\end{aligned} \tag{3.5}$$

And we find $\operatorname{rank}(\Sigma) = 266642187$, $\operatorname{rank}(\mathrm{T}) = 29804164155016$ according to this recipe.

Consequently, the rank of (2.2) turns out to be

$$\begin{aligned}&\textstyle\sum_{k=0}^{8} k!^2 \left\{{17 \atop k+1}\right\}\left\{{21 \atop k+1}\right\} + \big((81577\left\{{17 \atop 10}\right\} + 266642187)9! + 187258\big)\left\{{21 \atop 10}\right\} + 29804164155016\\ &\quad = 779216462178113853893868778420146{8},\end{aligned} \tag{3.6}$$

because that parade has order 9. (It's about 0.167% of $B_{16,20} = 466969543193729892903725378950448850{2}$.)

Now let's try *un*ranking. We know from (0.1) that $B_{4,7} = 1315666$. What is the millionth parade that can be formed with 4 girls and 7 boys? In other words, what parade of $\mathcal{P}_{4,7}$ has rank 999999, according to the ranking scheme that we've just discussed?

The parades of order d are enumerated by the term for $k = d$ in (1.11); and the numerical values of those terms when $(m,n) = (4,7)$ are respectively 1, 1905, 96600, 612360, 604800, for $d = 0$, 1, 2, 3, and 4.

Thus $1 + 1905 + 96600 + 612360 = 710866$ is the number of parades of orders 3 or less. But adding another 604800 will take us over a million. So, in accordance with (3.1), we seek the parade that has rank $999999 - 710866 = 289133$ in the set of order-4 parades, of which there are $4!\left\{{5 \atop 5}\right\}4!\left\{{8 \atop 5}\right\} = 604800$. In the latter formula, $4!\left\{{5 \atop 5}\right\}$ enumerates the possibilities for the ordered partition $S_0S_1S_2S_3S_4$ of the girls, and $4!\left\{{8 \atop 5}\right\}$ enumerates the possibilities for the ordered partition $T_1T_2T_3T_4T_5$ of the boys. Numerically, $4! = 24$, $\left\{{5 \atop 5}\right\} = 1$, and $\left\{{8 \atop 5}\right\} = 1050$.

To get the rank-289133 pattern from $24 \cdot 1 \cdot 24 \cdot 1050$ possibilities in accordance with (3.2), we have

$$289133 \;=\; ((11 \cdot 1 + 0) \cdot 24 + 11) \cdot 1050 + 383, \tag{3.7}$$

using a mixed-radix system with radices 24, 1, 24, and 1050.

The millionth parade will have the form $S_0T_1S_1T_2S_2T_3S_3T_4S_4T_5$, according to our interpretation above. Formula (3.7) tells us that the ordering of $\{S_1, \ldots, S_4\}$ should be the rank-11 permutation of $\{1,2,3,4\}$; those S's should come from the rank-0 set partition of 5 girls into 5 parts; the ordering of $\{T_1, T_2, T_3, T_4\}$ should (by coincidence) be the rank-11 permutation of $\{1,2,3,4\}$; and those T's should come from the rank-383 set partition of 8 boys into 5 parts.

In general, to get the rank-r permutation of $\{1,2,\ldots,n\}$, in lexicographic order, the recurrence $n! = n(n-1)!$ leads to the following algorithm: "First set $C_{n+1-j} \leftarrow r \bmod j$ and $r \leftarrow \lfloor r/j \rfloor$, for $j = 1, 2, \ldots, n$. Then, for j *decreasing* from n to 1, set $p_j \leftarrow 1 + C_j$, and increase p_i by 1 for all $i > j$ with $p_i \geq p_j$." For example, when $r = 11$ and $n = 4$, we find $C_1C_2C_3C_4 = 1210$, and the permutation $p_1p_2p_3p_4$ turns out to be 2431.

How should we unrank set partitions into a given number of blocks? Rule (3.5) has an equally slick counterpart, which finds the restricted growth string $a_0a_1\ldots a_n$ that has a given rank r and a given maximal element d:

$$\begin{aligned}&\text{"Set } i \leftarrow d\text{, and do the following for } j = n, n-1 \ldots, 0\text{:}\\&\text{If } r < (i+1)\left\{{j \atop i+1}\right\}\text{, set } a_j = \left\lfloor r/\left\{{j \atop i+1}\right\}\right\rfloor \text{ and } r \leftarrow r \bmod \left\{{j \atop i+1}\right\};\\&\text{otherwise set } a_j \leftarrow i,\ r \leftarrow r - (i+1)\left\{{j \atop i+1}\right\}\text{, and } i \leftarrow i-1\text{."}\end{aligned} \tag{3,8}$$

In particular, from the $\left\{{5 \atop 5}\right\} = 1$ set partitions when $m = 4$ and $d = 4$, we want the one for $r = 0$, which has the restricted growth string $a_0a_1a_2a_3a_4 = 01234$. (Hey, algorithms have to work in trivial cases too.) We apply the permutation 2431 to this, obtaining 02431; that means $G_0 = \{g_0\}$, $G_1 = \{g_4\}$, $G_2 = \{g_1\}$, $G_3 = \{g_3\}$, and $G_4 = \{g_2\}$, except that we're supposed to remove the "dummy" girl g_0 from G_0.

Proceeding similarly for the boys, the set partition of rank 383 when $n = 7$ and $d = 4$ turns out to have the restricted growth string 01123242. Apply the boys' permutation 2431, to get the digit string 02243414; hence $T_1 = \{b_6\}$, $T_2 = \{b_1, b_2\}$, $T_3 = \{b_4\}$, $T_4 = \{b_3, b_5\}$, $T_5 = \{b_0\}$; and b_0 is removed from T_5, which is the special set T_{d+1}.

Ta da: The millionth parade with four girls and seven boys is defined by $S_0T_1S_1T_2S_2T_3S_3T_4S_4T_5$, so it is

$$b_6g_4b_1b_2g_1b_4g_3b_3b_5b_7g_2\,. \tag{3.9}$$

Unfortunately, this example doesn't illustrate the general case in which S_0 and/or T_{d+1} are nonempty. We do get an example with nonempty S_0 if we interchange m and n, asking instead for the millionth parade when $m = 7$ and $n = 4$. That one turns out to be

$$g_5b_1g_6b_4g_1b_3g_3g_7b_2g_2g_4\,; \tag{3.10}$$

it comes from the partition of rank 0 and the permutation of rank 5 for the boys, together with the partition of rank 497 and the permutation of rank 11 for the girls.

The main advantage of a bijection between a combinatorial class $\mathcal{A}$ and the integers $[0\,.\,.\,|\mathcal{A}|)$ is that it allows us to generate *uniformly random patterns* from $\mathcal{A}$. For example, given a random number in the interval $[0\,.\,.\,B_{m,n})$, we can compute the corresponding parade by performing $O((m+n)\log(m+n))$ arithmetic operations on numbers that have $O((m+n)\log(m+n))$ bits, using the procedure above.

It's easy to understand the very first parade, according to this bijection: We set $d = 0$ and use the all-0 restricted growth strings and the empty permutation for both girls and boys. The result is $g_1g_2\ldots g_mb_1b_2\ldots b_n$.

Similarly, the very last parade is only slightly more difficult to describe. Suppose $m \leq n$. Then the construction yields $d = m$; the restricted growth strings are $012\ldots d$ for the girls and $0^{n-m+1}12\ldots d$ for the boys; both permutations are $d\ldots 21$. Hence the final parade is $b_ng_mb_{n-1}g_{m-1}\ldots b_{n+1-m}g_1b_1\ldots b_{n-m}$. When $m > n$, it is $g_1\ldots g_{m-n}b_ng_mb_{n-1}g_{m-1}\ldots b_1g_{m+1-n}$.

Notice that the ordering of the 14 parades in (2.1) does *not* correspond to our bijection. (They appear there in lexicographic order, assuming that $g_1 < g_2 < b_1 < b_2$.) If we had listed them in the order of our bijection, for ranks $r = 0, 1, \ldots, 13$, the respective values of d would have been (0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2). That listing would in fact have been

$$\begin{aligned}&g_1g_2b_1b_2,\ g_2b_1g_1b_2,\ g_2b_1b_2g_1,\ g_2b_2g_1b_1,\ b_1g_1g_2b_2,\ b_1b_2g_1g_2,\ b_2g_1g_2b_1,\\&\quad g_1b_1g_2b_2,\ g_1b_1b_2g_2,\ g_1b_2g_2b_1,\ b_1g_1b_2g_2,\ b_2g_1b_1g_2,\ b_1g_2b_2g_1,\ b_2g_2b_1g_1.\end{aligned} \tag{3.11}$$

Every parade with 1 girl and n boys corresponds naturally to the subset of boys that precedes the girl. When those parades are listed in order of rank, the subsets arise in the somewhat bizarre order

$$\begin{aligned}&\emptyset,\ \{b_1\},\ \{b_1b_2\},\ \{b_2\},\ \{b_1b_3\},\ \{b_1b_2b_3\},\ \{b_2b_3\},\ \{b_3\},\\&\quad\{b_1b_4\},\ \{b_1b_2b_4\},\ \{b_2b_4\},\ \{b_1b_3b_4\},\ \{b_1b_2b_3b_4\},\ \{b_2b_3b_4\},\ \{b_3b_4\},\ \{b_4\},\end{aligned} \tag{3.12}$$

which corresponds to the permutation $f(r)$ of positive integers defined by

$$f(2^k + j) = 2^k + f(j+1) \text{ for } 0 \leq j < 2^k - 1 \text{ and } f(2^{k+1} - 1) = 2^k, \quad \text{for all } k > 0. \tag{3.13}$$

The parades for m girls and 1 boy arise in essentially the same order, but with respect to the subsets of girls that *follow* the boy: $\emptyset$, $\{g_1\}$, $\{g_1g_2\}$, $\{g_2\}$, $\{g_1, g_3\}$, etc.

4. A recurrence for pB numbers. Taking a different tack, we can use the elegant bivariate generating function $G(w,z)$ in (1.7) to deduce an unexpected relationship between the numbers on adjacent rows of (0.1). Notice first that differentiation with respect to w has a simple "shift-up" effect on the coefficients:

$$\frac{\partial}{\partial w}G(w,z)=\frac{\partial}{\partial w}\sum_{m,n\ge 0}B_{m,n}\frac{w^m}{m!}\frac{z^n}{n!}=\sum_{m,n\ge 0}B_{m,n}\frac{w^{m-1}}{(m-1)!}\frac{z^n}{n!}=\sum_{m,n\ge 0}B_{m+1,n}\frac{w^m}{m!}\frac{z^n}{n!}. \qquad (4.1)$$

(In this formula, $1/(-1)!=0$.) Similarly, $\frac{\partial}{\partial z}G(w,z)=\sum_{m,n\ge 0}B_{m,n+1}\frac{w^m}{m!}\frac{z^n}{n!}$. Furthermore

$$\frac{\partial}{\partial w}\frac{e^{w+z}}{e^w+e^z-e^{w+z}}=\frac{e^{w+2z}}{(e^w+e^z-e^{w+z})^2}=e^{-w}G(w,z)^2. \qquad (4.2)$$

We also have

$$G(w,z)=\frac{1}{e^{-z}+e^{-w}-1}; \qquad (4.3)$$

whence it follows that

$$e^{-z}G(w,z)^2+e^{-w}G(w,z)^2-G(w,z)\;=\;G(w,z)^2. \qquad (4.4)$$

By equating the coefficients of $w^m z^n$ on both sides of this equation, we see that

$$B_{m,n+1}+B_{m+1,n}-B_{m,n}\;=\;\binom{n}{0}B_{m,n+1}+\binom{n}{1}B_{m,n}+\binom{n}{2}B_{m,n-1}+\cdots+\binom{n}{n}B_{m,1}. \qquad (4.5)$$

Therefore the pB numbers can be computed row by row, using a recurrence that's totally different from any of our previous formulas:

$$B_{0,n}=1;\quad B_{m+1,n}\;=\;B_{m,n}+\binom{n}{1}B_{m,n}+\binom{n}{2}B_{m,n-1}+\cdots+\binom{n}{n}B_{m,1}, \qquad (4.6)$$

valid for all $m,n\ge 0$. For example,

$$B_{3,4}=B_{2,4}+\tbinom{4}{1}B_{2,4}+\tbinom{4}{2}B_{2,3}+\tbinom{4}{3}B_{2,2}+\tbinom{4}{4}B_{2,1}=146+4\cdot 146+6\cdot 46+4\cdot 14+1\cdot 4=1066. \qquad (4.7)$$

(Masanobu Kaneko [8] derived the recurrence (4.6) shortly after he had discovered the pB numbers.)

OK, we know from algebra and calculus that the number of parades of girls and boys satisfies the recurrence (4.6). Can we also find a purely *combinatorial* explanation for that fact?

Yes! There's obviously only one possible parade when no girls are present; hence $B_{0,n}=1$. So suppose we have a parade Π with $m+1$ girls and n boys; we want to represent it uniquely as one of the parades represented by the right-hand side of (4.6). We'll say that $\Pi\in\mathcal{P}_{m+1,n}$ is of type T if T is the block of boys that immediately follows the oldest girl, g_{m+1}. For example, if $m=3$, the parade in (3.9) has type $\{b_1b_2\}$.

The number of parades of type $\emptyset$ is $B_{m,n}$, because such parades arise if and only if g_{m+1} comes last; conversely, g_{m+1} can safely be appended to any parade that has m girls.

Otherwise we shall show that the number of parades of type T is $B_{m,n+1-|T|}$; and this will establish (4.6), because there are $\binom{n}{t}$ types T with $|T|=t$. Let b_μ be the oldest boy in T. Remove $T\setminus\{b_\mu\}$ from the set of boys, and give the remaining boys the new names $b'_1,\ldots,b'_{n-(t-1)}$ (youngest to oldest). Then map $\Pi\mapsto\Pi'$ by renaming the boys, and by replacing the subsequence '$g_{m+1}T$' of Π by $b'_{\mu-(t-1)}$ (the new name of b_μ).

We've thereby mapped every $(m+1,n)$-parade of type $T\ne\emptyset$ into an $(m,n+1-|T|)$-parade; and the mapping is clearly invertible. For example, if $T=\{b_2b_3b_6\}$, the parade $\Pi'=b'_1b'_4b'_5g_1g_4b'_3g_2b'_2g_5b'_6g_3$ could have come only from $\Pi=b_1b_7g_6b_2b_3b_6g_1g_4b_5g_2b_4g_5b_8g_3$. (Further explanation is below.)

5. A recursive ranking scheme. Now let's turn the tables and assign ranks to objects that satisfy (4.6), instead of assigning ranks to objects that are enumerated by (1.11) as we did in §3. Once again, every parade in $\mathcal{P}_{m,n}$ will be assigned a number between 0 and $B_{m,n}-1$ inclusive.

In accordance with the additive rule (3.1), we'll give the smallest $B_{m,n}$ ranks to parades of type $\emptyset$. The next $\binom{n}{1}B_{m,n}$ ranks in (4.6) will go to the parades enumerated by $\binom{n}{1}B_{m,n}$; and so on. In general the right-hand side of (4.6) has $n+1$ terms, $t_0+t_1+t_2+\cdots+t_n$, where $t_0=B_{m,n}$ and $t_k=\binom{n}{k}B_{m,n+1-k}$ when $k>0$.

If we're unranking, the parade of rank r will belong to term t_k, where k is found as follows: "Set $k \leftarrow 0$. While $r \geq t_k$, set $r \leftarrow r - t_k$ and $k \leftarrow k+1$."

But let's do ranking first. What is the recursive rank of the parade $\Pi = b_6 g_4 b_1 b_2 g_1 b_4 g_3 b_3 b_5 b_7 g_2$ in (3.9), whose rank was 999999 under the old scheme? This parade of type $\{b_1 b_2\}$ is mapped into

$$\begin{aligned}\Pi' &= b_1 b_5 g_1 b_3 g_3 b_2 b_4 b_6 g_2 \text{ of type } \{b_2 b_4 b_6\}, \text{ which is mapped into}\\ \Pi'' &= b_1 b_3 g_1 b_2 b_4 g_2 \text{ of type } \emptyset, \text{ which is mapped into}\\ \Pi''' &= b_1 b_3 g_1 b_2 b_4 \text{ of type } \{b_2 b_4\}, \text{ which is mapped into}\\ \Pi'''' &= b_1 b_2 b_3.\end{aligned} \tag{5.1}$$

Consequently

$$\begin{aligned}\operatorname{rank}(\Pi) &= B_{3,7} + \tbinom{7}{1} B_{3,7} + 0B_{3,6} + \operatorname{rank}(\Pi');\\ \operatorname{rank}(\Pi') &= B_{2,6} + \tbinom{6}{1} B_{2,6} + \tbinom{6}{2} B_{2,5} + 14B_{2,4} + \operatorname{rank}(\Pi'');\\ \operatorname{rank}(\Pi'') &= \operatorname{rank}(\Pi''');\\ \operatorname{rank}(\Pi''') &= B_{0,4} + \tbinom{4}{1} B_{0,4} + 4B_{0,3} + \operatorname{rank}(\Pi'''');\\ \operatorname{rank}(\Pi'''') &= 0.\end{aligned} \tag{5.2}$$

So $\operatorname{rank}(\Pi''') = 9$, $\operatorname{rank}(\Pi'') = 9$, $\operatorname{rank}(\Pi') = 18621$, and $\operatorname{rank}(\Pi) = 701101$. (The coefficients in '$0B_{3,6}$', '$14B_{2,4}$', '$4B_{0,3}$' come from ranking the types: The rank of $\{b_1 b_2\}$ among 2-subsets of $\{b_1, \ldots, b_7\}$ is 0; the rank of $\{b_2 b_4 b_6\}$ among 3-subsets of $\{b_1, \ldots, b_6\}$ is 14; the rank of $\{b_2 b_4\}$ among 2-subsets of $\{b_1, \ldots, b_4\}$ is 4. Formula (A.2) in the Appendix below explains how such ranks are readily computed.)

When the same method is applied to the "typical" parade (2.2), which is of type $\{b_{10}\}$, we find that Π' has type $\{b_1 b_{12} b_{19}\}$, Π'' has type $\{b_{15}\}$ (where b_{15} was originally b_{17}), and so on. The recursive rank turns out to be 1491392338417882718739839722665904161, about 32% of $B_{16,20}$. Middle of the road.

The recursive *un*ranking procedure is another good test of these methods, so let's study it next. What is the millionth element of $\mathcal{P}_{4,7}$ according to this *new* ranking scheme? For that problem we have $m = 3$, $n = 7$, and the terms $(t_0, \ldots, t_7)$ are (85310, 597170, 425586, 165130, 37310, 4830, 322, 8). Hence the parade of rank $r = 999999$ leads to $k = 2$; it will be the parade of rank $999999 - 85310 - 597170 = 317519$ that corresponds to term $t_2 = \binom{7}{2} B_{3,6}$.

In accordance with the multiplicative rule (3.2), we now find $317519 = 15 \cdot B_{3,6} + 13529$. Algorithm (A.1) in the Appendix below tells us that the 2-subset of $\{b_1, \ldots, b_7\}$ that has rank 15 is $\{b_1 b_7\}$. So we want the parade of $\mathcal{P}_{3,6}$ that has type $\{b_1 b_7\}$ and rank 13529.

Let $\Pi_{m,n,r}$ be the parade of recursive rank r in $\mathcal{P}_{m,n}$. We've just concluded that $\Pi_{4,7,999999}$ is the parade of type $\{b_1 b_7\}$ that maps to $\Pi_{3,6,13529}$; we shall say that "$\Pi_{4,7,999999} = \Pi_{3,6,13529}$ extended by $\{b_1 b_7\}$."

It turns out that, similarly,

$$\begin{aligned}\Pi_{3,6,13529} &= \Pi_{2,5,139} \text{ extended by } \{b_3 b_5\};\\ \Pi_{2,5,139} &= \Pi_{1,5,11} \text{ extended by } \{b_4\};\\ \Pi_{1,5,11} &= \Pi_{0,4,0} \text{ extended by } \{b_3 b_4\}.\end{aligned} \tag{5.3}$$

Now $\Pi_{0,4,0}$ is $b_1 b_2 b_3 b_4$. So we can go backward in (5.3) and determine

$$\begin{aligned}\Pi_{1,5,11} &= b_1 b_2 b_5 g_1 b_3 b_4\,;\\ \Pi_{2,5,139} &= b_1 b_2 b_5 g_1 b_3 g_2 b_4\,;\\ \Pi_{3,6,13529} &= b_1 b_2 b_6 g_1 b_4 g_2 g_3 b_3 b_5\,;\\ \Pi_{4,7,999999} &= b_2 b_3 g_4 b_1 b_7 g_1 b_5 g_2 g_3 b_4 b_6\,.\end{aligned} \tag{5.4}$$

The algorithm that leads from (5.3) to (5.4), which is somewhat delicate, is described in the Appendix below. Incidentally, when girls and boys are reversed in this example, we find

$$\Pi_{7,4,999999} = g_3 b_2 b_3 g_2 g_7 b_4 g_1 g_5 g_6 b_1 g_4\,. \tag{5.5}$$

(That computation involves extending $\Pi_{3,4,847}$ by $\emptyset$.)

Notice that this procedure leads to an interesting characterization: *Every parade can be built up uniquely by starting with a parade that has no girls and repeatedly extending it, adding one girl at a time. Each nonempty extension from n' boys to n boys is guided by an $(n-n'+1)$-element subset of $\{b_1,\ldots,b_n\}$.*

The fourteen parades of $\mathcal{P}_{2,2}$ are ranked in the following order by this recursive scheme, in contrast to (2.1) and (3.11):

$$\begin{gathered} b_1b_2g_1g_2,\ b_2g_1b_1g_2,\ b_1g_1b_2g_2,\ g_1b_1b_2g_2,\ b_2g_2b_1g_1,\ b_2g_1g_2b_1,\ g_2b_1g_1b_2, \\ g_1b_2g_2b_1,\ b_1g_2b_2g_1,\ g_2b_2g_1b_1,\ b_1g_1g_2b_2,\ g_1b_1g_2b_2,\ g_2b_1b_2g_1,\ g_1g_2b_1b_2, \end{gathered} \tag{5.6}$$

In general, the first parade in this recursive ranking of $\mathcal{P}_{m,n}$ is clearly $b_1b_2\ldots b_ng_1g_2\ldots g_m$, because we start with all the boys and append all the girls, one by one.

The very last parade, on the other hand, is obtained when we extend the last parade of $\mathcal{P}_{m-1,1}$ by $\{b_1,\ldots,b_n\}$. So it is $g_1g_2\ldots g_mb_1b_2\ldots b_n$.

When the 2^n parades of $\mathcal{P}_{1,n}$ are ranked recursively, the subsets of boys that follow the girl appear in increasing order of their size, and in colexicographic order within each size. For example, the eight parades of $\mathcal{P}_{1,3}$ are

$$b_1b_2b_3g_1,\ b_2b_3g_1b_1,\ b_1b_3g_1b_2,\ b_1b_2g_1b_3,\ b_3g_1b_1b_2,\ b_2g_1b_1b_3,\ b_1g_1b_2b_3,\ g_1b_1b_2b_3. \tag{5.7}$$

It's the same as the order of subsets made by the girls who *precede* the boy in $\mathcal{P}_{n,1}$; for example,

$$b_1g_1g_2g_3,\ g_1b_1g_2g_3,\ g_2b_1g_1g_3,\ g_1g_2b_1g_3,\ g_3b_1g_1g_2,\ g_1g_3b_1g_2,\ g_2g_3b_1g_1,\ g_1g_2g_3b_1. \tag{5.8}$$

6. Automorphisms. There are many one-to-one mappings of $\mathcal{P}_{m,n}$ into itself; in fact, the total number is $B_{m,n}!$, which is huge. The vast majority of them are completely arbitrary and of no interest whatsoever. But some of them are particularly important, because they're "natural" and easy to understand. Indeed, we obtain $m!\,n!$ natural automorphisms by using any permutation $\sigma_1\ldots\sigma_m$ of $[1\,..\,m]$ to rename the girls and any permutation $\tau_1\ldots\tau_n$ of $[1\,..\,n]$ to rename the boys; after replacing g_j by g_{σ_j} and b_k by b_{τ_k}, for all j and k, adjacent girls and adjacent boys can bubblesort themselves and form a new parade.

For example, suppose $m=4$, $n=7$, $\sigma_1\ldots\sigma_4=3142$, and $\tau_1\ldots\tau_7=5721643$. Then the parade $b_6g_4b_1b_2g_1b_4g_3b_3b_5b_7g_2$ of (3.9) becomes $b_4g_2b_5b_7g_3b_1g_4b_2b_6b_3g_1$ before sorting, and $b_4g_2b_5b_7g_3b_1g_4b_2b_3b_6g_1$ afterwards.

Another straightforward automorphism reflects the parade, left-to-right. Then (3.9) becomes the pseudo-parade $g_2b_7b_5b_3g_3b_4g_1b_2b_1g_4b_6$ before sorting, and $g_2b_3b_5b_7g_3b_4g_1b_1b_2g_4b_6$ afterwards.

Furthermore, the number of basic automorphisms increases by a further factor of $2!^2 3!^2 \ldots \min\{m,n\}!^2$ when we realize that we can specify, for each k, two independent permutations of $[1\,..\,k]$ that can be applied respectively to the blocks $S_1\ldots S_k$ and $T_1\ldots T_k$ that occur in parades of order k.

This multiplicity of automorphisms means that every bijection between $\mathcal{P}_{m,n}$ and another class of combinatorial patterns leads to many further bijections. Some of those bijections will, of course, be much more intuitive and/or interesting than others.

7. Acyclic bipartite orientations. It's high time to fulfill the promise that was made in the introduction, namely to discuss other combinatorial patterns that are enumerated by the pB numbers.

Let $\mathcal{O}_{m,n}$ be the set of all ways to assign a direction to each of the mn edges of the complete bipartite graph $K_{m,n}$, in such a way that the resulting digraph has no oriented cycles.

It turns out that $|\mathcal{O}_{m,n}|=B_{m,n}$. For example, when $m=n=2$, the four edges $u_1 \longrightarrow v_1 \longrightarrow u_2 \longrightarrow v_2 \longrightarrow u_1$ of $K_{2,2}$ form a 4-cycle; and exactly two of the 16 ways to orient those edges will turn that cycle into an oriented cycle. That leaves $14=B_{2,2}$ acyclic orientations.

Indeed, there's an appealing bijection between $\mathcal{P}_{m,n}$ and $\mathcal{O}_{m,n}$: We can associate the vertices of $K_{m,n}$ with m girls in one part and n boys in the other. Given any parade in $\mathcal{P}_{m,n}$, we orient the edge $g_i \,—\, b_j$ by simply saying that $g_i \longrightarrow b_j$ if and only if b_j follows g_i in the parade. The resulting digraph is obviously acyclic.

Conversely, this mapping from $\mathcal{P}_{m,n}$ to $\mathcal{O}_{m,n}$ is invertible. Given any acyclic orientation, we want to show that it corresponds to a unique parade. At least one vertex must be a "source," having no predecessor; and the sources must either be all girls or all boys. Place all the sources first in the parade; remove them from the digraph; and repeat the argument.

Any orientation of $K_{m,n}$ can be represented conveniently as an $m \times n$ matrix a_{ij} of 0s and 1s, where $a_{ij} = 1$ if and only if $u_i \longrightarrow v_j$. For example, the acyclic orientation of $K_{4,7}$ that corresponds under our bijection to (3.9), the millionth parade of $\mathcal{P}_{4,7}$, is

$$\begin{pmatrix} 0&0&1&1&1&0&1\\ 0&0&0&0&0&0&0\\ 0&0&1&0&1&0&1\\ 1&1&1&1&1&0&1 \end{pmatrix}. \tag{7.1}$$

It's easy to reconstruct (3.9) from this. *Hint:* A girl source is a row of 1s; a boy source is a column of 0s.

(The fact that $|\mathcal{O}_{m,n}| = B_{m,n}$ was discovered by Peter Cameron, Celia Glass, Kamilla Rekvényi, and Robert Schumacher [12].)

8. Doubly bounded permutations. Let $\mathcal{V}_{m,n}$ be the set of all permutations $p_1p_2\ldots p_{m+n}$ of $[1\,.\,.\,m+n]$ with the property that

$$j - m \le p_j \le j + n \qquad \text{for } 1 \le j \le m+n. \tag{8.1}$$

Guess what? $|\mathcal{V}_{m,n}| = B_{m,n}$. For example, when $m = n = 2$, the permutation $p_1p_2p_3p_4$ must have $p_1 \ne 4$ and $p_4 \ne 1$. Of the 24 possibilities, we must throw out the 6 with $p_1 = 4$ and the 6 with $p_4 = 1$; but we threw 4231 and 4321 out twice, so exactly $14 = B_{2,2}$ remain.

The number of permutations that we seek is the *permanent* of a nicely structured $(m+n) \times (m+n)$ matrix $Q_{m,n}$, illustrated here for $m = 4$ and $n = 7$:

$$Q_{m,n} = \begin{pmatrix} 1&1&1&1&1&1&1&1&0&0&0\\ 1&1&1&1&1&1&1&1&1&0&0\\ 1&1&1&1&1&1&1&1&1&1&0\\ 1&1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1&1\\ 0&1&1&1&1&1&1&1&1&1&1\\ 0&0&1&1&1&1&1&1&1&1&1\\ 0&0&0&1&1&1&1&1&1&1&1\\ 0&0&0&0&1&1&1&1&1&1&1\\ 0&0&0&0&0&1&1&1&1&1&1\\ 0&0&0&0&0&0&1&1&1&1&1 \end{pmatrix} = \begin{pmatrix} J_{m,n} & S_m \\ S_n^T & J_{n,m} \end{pmatrix}. \tag{8.2}$$

Here $J_{m,n}$ is an $m \times n$ matrix of all 1s; S_m is a lower-triangular $m \times m$ matrix with 1s on and below the diagonal, but with 0s above. The value of $|\mathcal{V}_{m,n}| = \mathrm{per}(Q_{m,n})$ is the number of ways we can place rooks on the 1s of $Q_{m,n}$, with no two rooks in the same row or the same column.

Suppose we place k rooks in $J_{m,n}$, the submatrix at the upper left. Then $m-k$ rooks must be placed in S_m, and $n-k$ rooks in S_n^T. Consequently there are k rooks also in $J_{n,m}$.

And now — aha — the number of ways to place $m-k$ nonattacking rooks on the 1s of S_m is exactly the quantity ${m+1 \brace k+1}$ that appears in formula (1.1)! This remarkable fact, discovered by Irving Kaplansky and John Riordan in 1946 [28], comes to us accompanied by a splendid *bijection* between the restricted growth strings $a_0a_1\ldots a_m$ with maximum element k and the placements of $m-k$ rooks, discovered by Edward Bender in 1969: If $j > 0$ and a_j exceeds $\max\{a_0,\ldots,a_{j-1}\}$, we put no rook into row j; otherwise we place a rook in column $i+1$ of row j, where $i < j$ is maximum such that $a_i = a_j$. (See exercise 5.1.3–19 in [3].)

Bender's bijection is illustrated for $m = 3$ and $k = 1$ in the following seven cases, where each possibility for the rooks is shown above its corresponding restricted growth string:

$$\begin{matrix}\begin{pmatrix} \text{♜}&0&0\\ 1&\text{♜}&0\\ 1&1&1 \end{pmatrix}\\ 0001\end{matrix} \quad \begin{matrix}\begin{pmatrix} \text{♜}&0&0\\ 1&1&0\\ 1&\text{♜}&1 \end{pmatrix}\\ 0010\end{matrix} \quad \begin{matrix}\begin{pmatrix} \text{♜}&0&0\\ 1&1&0\\ 1&1&\text{♜} \end{pmatrix}\\ 0011\end{matrix} \quad \begin{matrix}\begin{pmatrix} 1&0&0\\ \text{♜}&1&0\\ 1&1&\text{♜} \end{pmatrix}\\ 0100\end{matrix} \quad \begin{matrix}\begin{pmatrix} 1&0&0\\ \text{♜}&1&0\\ 1&\text{♜}&1 \end{pmatrix}\\ 0101\end{matrix} \quad \begin{matrix}\begin{pmatrix} 1&0&0\\ 1&\text{♜}&0\\ \text{♜}&1&1 \end{pmatrix}\\ 0110\end{matrix} \quad \begin{matrix}\begin{pmatrix} 1&0&0\\ 1&\text{♜}&0\\ 1&1&\text{♜} \end{pmatrix}\\ 0111\end{matrix}. \tag{8.3}$$

Once we've place the rooks into S_m and S_n^T, we're left with $k \times k$ submatrices of $J_{m,n}$ and $J_{n,m}$ where rooks can still be placed. Since the are $k!^2$ ways to complete the job, we've proved that $|\mathcal{V}_{m,n}|$ is the sum (1.11), which is $B_{m,n}$.

In fact, this argument also provides us with a simple bijection. For example, we know that the millionth parade (3.9) has the restricted growth sequence 01234 for the girls, together with the permutation 2431; and it has the restricted growth sequence 01123242 for the boys, together with the permutation 2431. Using Bender's bijection, and transposing the boys' placement in S_n in order to cover S_n^T, the corresponding rook placement in $Q_{4,7}$ turns out to be

$$\begin{pmatrix}
1 & 1 & \text{♜} & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0\\
1 & 1 & 1 & 1 & 1 & \text{♜} & 1 & 1 & 1 & 0 & 0\\
1 & 1 & 1 & \text{♜} & 1 & 1 & 1 & 1 & 1 & 1 & 0\\
\text{♜} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \text{♜} & 1 & 1\\
0 & \text{♜} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \text{♜}\\
0 & 0 & 0 & 1 & \text{♜} & 1 & 1 & 1 & 1 & 1 & 1\\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & \text{♜} & 1\\
0 & 0 & 0 & 0 & 0 & 1 & \text{♜} & 1 & 1 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \text{♜} & 1 & 1 & 1
\end{pmatrix}. \tag{8.4}$$

(Instead of transposing the boys' placement, we could have rotated it by 180°; it's unclear which alternative is better.) Given the placements in (8.4), the millionth doubly bounded permutation in $\mathcal{V}_{4,7}$ is

$$p_1 \ldots p_{11} \;=\; 3\;6\;4\;1\;9\;2\;11\;5\;10\;7\;8\,. \tag{8.5}$$

Exercise 8.1. What permutation $p_1 \ldots p_{36}$ of $\mathcal{V}_{16,20}$ corresponds to the "typical" parade (2.2) of $\mathcal{P}_{16,20}$?

(The number of permutations satisfying (8.1) was found in 1974 by Katalin Vesztergombi [17], who actually solved a much more general problem, as we shall see below. Stéphane Launois [13] noticed in 2007 that her formula matches (1.5); his paper was apparently the first publication to point out that pB numbers can have combinatorial significance. The bijection between $\mathcal{V}_{m,n}$ and $\mathcal{P}_{m,n}$ mentioned here is based on L. Lovász's solution to a similar problem; see [11], Problem 4.36.)

Notice, by the way, that the *inverses* of the permutations in $\mathcal{V}_{m,n}$ are the permutations in $\mathcal{V}_{n,m}$, because $Q_{m,n}^T = Q_{n,m}$.

9. Weak-excedance-first permutations. Let $\mathcal{E}_{m,n}$ be the set of all permutations $q_1 \ldots q_{m+n}$ for which (i) $q_j \ge j$ for $1 \le j \le m$, and (ii) $q_j \le j$ for $m < j \le m+n$. (Condition (i) is called a "weak excedance," in contrast to the condition '$q_j > j$', which is simply called an *excedance*. Condition (ii) is a "non-excedance.") For example, the elements of $\mathcal{E}_{2,2}$ are

$$1234, 1324, 1423, 1432, 2314, 2413, 2431, 3214, 3412, 3421, 4213, 4231, 4312, 4321. \tag{9.1}$$

Such permutations were introduced by Beáta Bényi and Péter Hajnal [25, §3.2].

We can count them by evaluating the permanent of a suitable $(m+n)\times(m+n)$ matrix, which looks like this in the special case $m=4$ and $n=7$:

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0\\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0\\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0\\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0\\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}. \tag{9.2}$$

Aha! It's just a jumbled-up version of $Q_{m,n}$ in (8.2), containing S_m^T, $J_{m,n}$, $J_{n,m}$, and S_n as submatrices.

Consequently every permutation $q_1 \ldots q_{m+n}$ of $\mathcal{E}_{m,n}$ is in bijection with the permutation

$$p_1 \ldots p_{m-1} p_m p_{m+1} \ldots p_{m+n-1} p_{m+n} \;=\; \bar{q}_m \ldots \bar{q}_2 \bar{q}_1 \bar{q}_{m+n} \ldots \bar{q}_{m+2} \bar{q}_{m+1} \tag{9.3}$$

of $\mathcal{V}_{m,n}$, where $\bar{q} = m + n + 1 - q$. By (8.5), the millionth weak-excedance-first permutation in $\mathcal{E}_{4,7}$ is

$$q_1 \ldots q_{11} \;=\; 11\ 8\ 6\ 9\ 4\ 5\ 2\ 7\ 1\ 10\ 3\,. \tag{9.4}$$

10. Lonesum matrices. The row sums $R = (r_1, \ldots, r_m)$ and column sums $S = (s_1, \ldots, s_n)$ of an $m \times n$ matrix (a_{ij}), namely

$$r_i \;=\; \sum_{j=1}^{n} a_{ij} \qquad \text{and} \qquad s_j \;=\; \sum_{i=1}^{m} a_{ij}\,, \tag{10.1}$$

are important in many applications, especially when all of the matrix entries a_{ij} are 0 or 1. Herbert Ryser [19] found necessary and sufficient conditions for the existence of at least one 0–1 matrix whose row and column sums match a given pair (R, S). He also showed that any two 0–1 matrices with the *same* (R, S) can be transformed into each other by a sequence of "switches," where every switch changes a 2×2 submatrix from $\binom{1\,0}{0\,1}$ to $\binom{0\,1}{1\,0}$ or vice versa. (Such a switch clearly leaves all row and column sums unchanged.)

Therefore a 0–1 matrix is *uniquely* determined by its R and S sequences if and only if it is "$\{\binom{1\,0}{0\,1}, \binom{0\,1}{1\,0}\}$-free"; that is, if and only if none of its 2×2 submatrices have the forms $\binom{1\,0}{0\,1}$ or $\binom{0\,1}{1\,0}$. For example, the matrix (7.1) is $\{\binom{1\,0}{0\,1}, \binom{0\,1}{1\,0}\}$-free; so it's the only one with $R = (4, 0, 3, 6)$ and $S = (1, 1, 3, 2, 3, 0, 3)$.

Let $\mathcal{L}_{m,n}$ be the set of all $m \times n$ matrices of 0s and 1s whose row and column sums determine them uniquely. Chad Brewbaker [18], calling such matrices "lonesum," proved that $|\mathcal{L}_{m,n}| = B_{m,n}$. (His paper was the *second* publication that presented pB numbers in a combinatorial context.)

We have almost proved his theorem already, because it's easy to see that *an orientation of a complete bipartite graph is acyclic if and only if it has no oriented 4-cycle.* For if the shortest oriented cycle has length k, the value of k must be even; and we can assign labels to the vertices so that the cycle has the form

$$u_1 \longrightarrow v_1 \longrightarrow u_2 \longrightarrow v_2 \longrightarrow \cdots \longrightarrow u_{k/2} \longrightarrow v_{k/2} \longrightarrow u_1\,. \tag{10.2}$$

There's a contradiction if $k > 4$, because both $v_2 \longrightarrow u_1$ and $u_1 \longrightarrow v_2$ would give a shorter cycle.

It follows that an $m \times n$ matrix of 0s and 1 is lonesum if and only if it is one of the matrices such as (7.1) that describes an acyclic orientation of $K_{m,n}$. (An oriented 4-cycle $u_i \longrightarrow v_j \longrightarrow u_{i'} \longrightarrow v_{j'} \longrightarrow u_i$ would show up in the matrix as $\binom{1\,0}{0\,1}$ or $\binom{0\,1}{1\,0}$ in rows $\{i, i'\}$ and columns $\{j, j'\}$.)

The bijection we used for $\mathcal{O}_{m,n}$ therefore works also for $\mathcal{L}_{m,n}$. The $\{\binom{1\,0}{0\,1}, \binom{0\,1}{1\,0}\}$-free matrix (7.1) is the millionth lonesum matrix of $\mathcal{L}_{4,7}$. It is easily reconstructed from its row sums $(4, 0, 3, 6)$ and column sums $(1, 1, 3, 2, 3, 0, 3)$.

(It's not difficult to see that lonesum matrices are precisely the matrices that can be transformed by row and column permutation to the Ferrers diagram for an integer partition, with all the 1s concentrated at the top and left, because any permutation of R and/or S preserves lonesumness. In a Ferrers diagram, the row and column sums appear in nonincreasing order, and they're "conjugate" partitions of their sum.)

11. Strongly Γ-free matrices. A $\{\binom{1\,1}{1\,0}, \binom{1\,1}{1\,1}\}$-free matrix is called "strongly Γ-free," because $\binom{1\,1}{1\,0}$ looks like the letter Γ and $\binom{1\,1}{1\,1}$ is another matrix of the form $\binom{1\,1}{1\,*}$. We shall let $\mathcal{G}_{m,n}$ be the set of all $m \times n$ matrices of 0s and 1s that are strongly Γ-free, and (surprise?) we shall prove that $|\mathcal{G}_{m,n}| = B_{m,n}$. It's obvious that $|\mathcal{G}_{2,2}| = 14 = B_{2,2}$.

(The much larger class of $\binom{1\,1}{1\,0}$-free matrices, which was called simply "Γ-free" by Anna Lubiw [20], also has important applications to combinatorial optimization. There are $(n + 3)3^{n-1}$ Γ-free matrices when $m = 2$, and 1725320 of them when $m = n = 5$.)

The strong Γ-free constraint makes it easy to evaluate $|\mathcal{G}_{m,n}|$ by showing that recurrence (4.6) holds. Clearly $|\mathcal{G}_{0,n}| = 1$ and $|\mathcal{G}_{1,n}| = 2^n$, because a matrix with fewer than 2 rows has no 2×2 submatrices.

If we're given an arbitrary matrix of $\mathcal{G}_{m+1,n}$, suppose there are exactly t 1s in its top row. If $t=0$, the remaining m rows are a perfectly general matrix of $\mathcal{G}_{m,n}$. Otherwise the first $t-1$ of those 1s must have nothing but 0s below them; and if we remove those $t-1$ columns, we obtain a perfectly general matrix of $\mathcal{G}_{m,n+1-t}$. Recurrence (4.6) is valid because the 1s of the top row can appear in $\binom{n}{t}$ columns.

It's also easy to convert that argument to a bijection with parades. Let $\Gamma_{m,n,r}$ be the strongly Γ-free 0–1 matrix of rank r, computed according to the recurrence. We simply let $\Gamma_{m,n,r}$ correspond to $\Pi_{m,n,r}$, the parade of recursive rank r that was defined constructively in §5 above.

For example, the millionth matrix in $\mathcal{G}_{4,7}$ is $\Gamma_{4,7,999999}$, which corresponds to $\Pi_{4,7,999999}$. From (5.4), we have

$$\Gamma_{1,5,11} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \end{pmatrix}; \qquad \Gamma_{2,5,139} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}; \qquad \Gamma_{3,6,13529} = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix};$$

$$\Gamma_{4,7,999999} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}. \tag{11.1}$$

The boys that follow a particular girl in the parade correspond to 1s in a particular row of the matrix.

Conversely, we've seen that the millionth parade (3.9) has recursive rank 701101. So it corresponds to

$$\Gamma_{4,7,701101} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \tag{11.2}$$

by the calculations in (5.1) and (5.2). These Gamma-avoiding matrices follow the recursion (4.6) so closely, we can regard $\Gamma_{m,n,r}$ as a natural way to represent the parade $\Pi_{m,n,r}$.

Exercise 11.1. What parade corresponds to the following matrix of $\mathcal{G}_{4,7}$?

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Exercise 11.2. What strongly Γ-free 16×20 matrix corresponds to the "typical" parade Π of (2.2)?

Exercise 11.3. What's the maximum number of 1s in an element of $\mathcal{G}_{m,n}$?

(Bijections between $\mathcal{G}_{m,n}$ and parade-like arrangements were first constructed by Bényi and Hajnal [2], then simplified by Bényi and Nagy [4]; but those bijections were more complicated than the one above.)

Notice, by the way, that strongly Γ-free matrices are also in bijection with "strongly L-free matrices," namely the matrices that are $\{\binom{1\,0}{1\,1}, \binom{1\,1}{1\,1}\}$-free, under the obvious operation of top-to-bottom reflection. And of course there are similar bijections with matrices that are $\{\binom{0\,1}{1\,1}, \binom{1\,1}{1\,1}\}$-free, or $\{\binom{1\,1}{0\,1}, \binom{1\,1}{1\,1}\}$-free.

12. Γ-and-L-free matrices. Let $\mathcal{Q}_{m,n}$ be the set of all $m \times n$ matrices of 0s and 1s that are $\{\binom{1\,1}{1\,0}, \binom{1\,0}{1\,1}\}$-free. Chad Brewbaker, after completing his explorations of lonesum matrices, began to suspect that such matrices might be another pB class, and he communicated this question to Beáta Bényi and Péter Hajnal. They found [25] that, yes indeed, $|\mathcal{Q}_{m,n}| = B_{m,n}$, because the same recurrence, (4.6), is satisfied—but with m and n reversed.

Suppose a matrix of $\mathcal{Q}_{m,n+1}$ has an all-zero first column, or only one 1 in that column. Then its remaining columns can be any element of $\mathcal{Q}_{m,n}$. On the other hand, if the matrix has $t > 1$ rows that begin with 1, those rows must be identical; so there are $\binom{m}{t}$ times $|\mathcal{Q}_{m+1-t,n}|$ matrices for every such t.

That argument leads to a very simple bijection from $\mathcal{Q}_{m,n}$ to $\mathcal{G}_{m,n}$: We simply work from left to right. In any column with more than one 1, zero out the entries to the right of all but the bottommost 1. And to go back, go from right to left, copying entries from the right of the bottommost 1.

For example, the element of $\mathcal{Q}_{7,4}$ that corresponds to the transpose of the matrix $\Gamma_{4,7,999999}$ in (11.1) is

$$\begin{pmatrix} 1&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&1&1&1\\ 0&0&0&1\\ 0&1&1&1\\ 1&0&0&0 \end{pmatrix} \qquad \longleftrightarrow \qquad \begin{pmatrix} 1&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&1&0&0\\ 0&0&0&1\\ 0&1&1&1\\ 1&0&0&0 \end{pmatrix}. \tag{12.1}$$

(Notice that transposition is a bijection between $\mathcal{G}_{m,n}$ and $\mathcal{G}_{n,m}$, while top-to-bottom reflection is an automorphism of $\mathcal{Q}_{m,n}$.)

Of course $\mathcal{Q}_{m,n}$ is bijectively equivalent to $m \times n$ matrices that are $\{\binom{1\,1}{0\,1}, \binom{0\,1}{1\,1}\}$-free, $\{\binom{0\,0}{0\,1}, \binom{0\,1}{0\,0}\}$-free, or $\{\binom{0\,0}{1\,0}, \binom{1\,0}{0\,0}\}$-free, as well as to the $n \times m$ matrices that are $\{\binom{0\,1}{1\,1}, \binom{1\,0}{1\,1}\}$-free, $\{\binom{1\,1}{0\,1}, \binom{1\,1}{1\,0}\}$-free, $\{\binom{1\,0}{0\,0}, \binom{0\,1}{0\,0}\}$-free, or $\{\binom{0\,0}{1\,0}, \binom{0\,0}{0\,1}\}$-free.

13. Lonesum matrices redux. In §10 and §11, we've constructed bijections from $\mathcal{L}_{m,n}$ to $\mathcal{P}_{m,n}$ to $\mathcal{G}_{m,n}$. Thus, we know how to start with an $m \times n$ matrix that's lonesum and find a corresponding $m \times n$ matrix that's strongly Γ-free. It's instructive now to study the *composition* of those bijections, because the resulting process can be understood in terms of matrices alone, without reference to the intermediate parades that gave us the original insights.

Given a matrix $\Gamma \in \mathcal{G}_{m,n}$, we shall find a corresponding matrix $\Lambda \in \mathcal{L}_{m,n}$, where the correspondence is reversible. (In fact, if Γ happens to be $\Gamma_{m,n,r}$, which is the strongly Γ-free matrix of rank r, then Λ will be $\Lambda_{m,n,r}$, the matrix that corresponds via the bijection of §10 to the parade $\Pi_{m,n,r}$, which was defined in §5! But we won't need to "look under the hood" at that machinery, nor will we even need to know anything about parades when defining this bijection.)

To start, if $m = 1$ we simply let $\Lambda = \Gamma$. Suppose therefore that $m > 1$. Let Γ' be the bottom $m-1$ rows of Γ, and let Λ' be the matrix that corresponds to Γ'. We'll give a rule that tells how to obtain Λ by putting an appropriate new row above Λ', and by making a simple adjustment to Λ' itself.

The construction depends, of course, on the top row of Γ. If that top row is entirely zero, the top row of Λ will also be zero. Otherwise let Γ have 1s in columns $j_1 < \cdots < j_t$ of its top row. We know that columns $j_1, \ldots, j_{t-1}$ of Γ' will all be zero. (By induction, those columns of Λ' will also be zero.)

If column j_t of Λ' is all 1s, we simply let the top row of Λ be the top row of Γ. Otherwise let r_λ be the maximum row sum over all rows that have 0 in column j_t of Λ'; and let the top row of Λ be the top row of Γ *plus* row λ of Λ'.

Finally, modify Λ by changing columns $j_1, \ldots, j_{t-1}$ so that they all are copies of column j_t.

For example, the matrices Λ obtained from $\Gamma_{4,7,999999}$ and $\Gamma_{4,7,701101}$ in (11.1) and (11.2) are

$$\Lambda_{4,7,999999} = \begin{pmatrix} 1&0&0&1&1&1&1\\ 0&0&0&1&0&1&0\\ 0&0&0&1&0&1&0\\ 0&0&0&1&1&1&0 \end{pmatrix} \qquad \text{and} \qquad \Lambda_{4,7,701101} = \begin{pmatrix} 1&1&1&1&1&0&1\\ 0&0&1&0&1&0&1\\ 0&0&0&0&0&0&0\\ 0&0&1&1&1&0&1 \end{pmatrix}. \tag{13.1}$$

To go back from Λ to Γ, we just need to identify $j_1, \ldots, j_t$. Of all the columns with 1 in the top row, they're the ones whose column sum is smallest.

Exercise 13.1. What lonesum matrix corresponds to the plurisum matrix in exercise 11.1?

Exercise 13.2. What lonesum matrix corresponds to the "typical" matrix in the answer to exercise 11.2?

Exercise 13.3. What matrix Γ corresponds to Λ when Λ is a Ferrers diagram? (A Ferrers diagram has 1 in column j of row i if and only if $j \le p_i$, where $p_1 \ge \cdots \ge p_m$ is a given sequence of nonnegative integers.)

Exercise 13.4. For how many matrices does this bijection between $\mathcal{L}_{m,n}$ and $\mathcal{G}_{m,n}$ yield $\Lambda = \Gamma$?

Exercise 13.5. Let $\Lambda \in \mathcal{L}_{m,n}$ correspond to $\Gamma \in \mathcal{G}_{m,n}$ as above. True or false: (a) Row i of Λ is zero if and only if row i of Γ is zero. (b) Column j of Λ is zero if and only if column j of Γ is zero. (c) If matrices $\widehat{\Lambda}$ and $\widehat{\Gamma}$ are is obtained from Λ and Γ by deleting all of the zero rows and all of the zero columns, then $\widehat{\Lambda} = \widehat{\Gamma}$.

14. Max-closed relations. Our final example comes from yet another branch of discrete mathematics, the *binary relations* between two linearly ordered sets X and Y. Any relation '$\smile$' between $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ is equivalent to an $m \times n$ matrix, whose entry in row i and column j is 1 if x_i and y_j satisfy the relation (written '$x_i \smile y_j$'), but it's 0 if they do not ('$x_i \not\smile y_j$').

We assume that the elements are linearly ordered, with $x_1 < \cdots < x_m$ and $y_1 < \cdots < y_n$. The relation is called "max-closed" when it satisfies the condition

$$x_i \smile y_j \text{ and } x_{i'} \smile y_{j'} \text{ implies } x_{\max\{i,i'\}} \smile y_{\max\{j,j'\}}. \tag{14.1}$$

Max-closed relations were introduced in 1995 by Jeavons and Cooper [24], who observed that constraint satisfaction problems can be solved efficiently whenever they involve only max-closed constraints. (In the special case $m = n = 2$, a constraint satisfaction problem is a Boolean satisfiability problem, and max-closed constraints correspond to so-called "dual Horn clauses.")

Definition (14.1) puts us back into familiar territory, because it amounts to saying that a matrix defines a max-closed relation if and only if the matrix is $\{\binom{0\,1}{1\,0}, \binom{1\,1}{1\,0}\}$-free.

Let $\mathcal{M}_{m,n}$ be the set of all max-closed relations between ordered domains of sizes m and n. We shall prove that $|\mathcal{M}_{m,n}| = B_{m,n}$ by constructing a bijection between $\mathcal{M}_{m,n}$ and $\mathcal{G}_{m,n}$, as suggested by Ira Gessel.

Gessel's bijection is, in fact, amazingly simple, once you've seen it. Take any matrix in $\mathcal{M}_{m,n}$ and rotate it by 180°. This gives a $\{\binom{0\,1}{1\,0}, \binom{0\,1}{1\,1}\}$-free matrix (which is equivalent to a *min-closed* relation). Now repeatedly take any 2×2 submatrix that has the form $\binom{1\,1}{1\,*}$ and change it to $\binom{0\,1}{1\,*}$. The resulting matrix is strongly Γ-free(!).

(This transformation essentially works from left to right, looking at the bottommost 1 in each column and inserting 0s above it when needed. The inverse transformation also works from left to right and looks at each bottommost 1; but it inserts 1s when necessary.)

For example, the matrices of $\mathcal{G}_{4,7}$ in (11.1) and (11.2) correspond to the min-closed relations

$$\begin{pmatrix} 1&0&0&1&1&1&1\\ 0&0&0&1&1&1&0\\ 0&0&0&0&1&1&0\\ 0&0&0&0&1&1&0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1&1&0&0&0&0&0\\ 0&0&1&1&1&0&1\\ 0&0&0&0&0&0&0\\ 0&0&0&1&0&0&1 \end{pmatrix}, \tag{14.2}$$

which correspond, in turn, to the max-closed relations

$$\begin{pmatrix} 0&1&1&0&0&0&0\\ 0&1&1&0&0&0&0\\ 0&1&1&1&0&0&0\\ 1&1&1&1&0&0&1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1&0&0&1&0&0&0\\ 0&0&0&0&0&0&0\\ 1&0&1&1&1&0&0\\ 0&0&0&0&0&1&1 \end{pmatrix} \tag{14.3}$$

in $\mathcal{M}_{m,n}$.

Exercise 14.1. What max-closed 16×20 matrix corresponds to the "typical" parade Π of (2.2)?

Max-closed binary relations are also equivalent to another well-studied class of combinatorial patterns, called *permutation tableaux*, in the cases where the tableau is a rectangular matrix. In this context, Einar Steingrímsson and Lauren Williams have devised an interesting "zig-zag" bijection between $\mathcal{M}_{m,n}$ and $\mathcal{E}_{m,n}$; see [26] and the exposition in [3, exercise 5.1.4–45].

Temporary note to the reader: I shall revise the following sections soon, because there are much better ways to present the rest of the story. Don't bother to read further unless you are *really* curious!

15. Companion numbers. Two other symmetrical arrays of numbers turn out to be intimately related to the pB numbers (0.1), and they too play important roles in our story. They're called $C_{m,n}$ and $D_{m,n}$, and they look like this:

$C_{m,n}$	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
$m=0$	1	1	1	1	1	1	1	1
$m=1$	1	3	7	15	31	63	127	255
$m=2$	1	7	31	115	391	1267	3991	12355
$m=3$	1	15	115	675	3451	16275	72955	316275
$m=4$	1	31	391	3451	25231	164731	999391	5767051
$m=5$	1	63	1267	16275	164731	1441923	11467387	85314915
$m=6$	1	127	3991	72955	999391	11467387	116914351	1096832395
$m=7$	1	255	12355	316275	5767051	85314915	1096832395	12764590275

(15.1)

$D_{m,n}$	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
$m=0$	1	0	0	0	0	0	0	0
$m=1$	0	1	1	1	1	1	1	1
$m=2$	0	1	5	13	29	61	125	253
$m=3$	0	1	13	73	301	1081	3613	11593
$m=4$	0	1	29	301	2069	11581	57749	268381
$m=5$	0	1	61	1081	11581	95401	673261	4306681
$m=6$	0	1	125	3613	57749	673261	6487445	55213453
$m=7$	0	1	253	11593	268381	4306681	55213453	610093513

(15.2)

Their bivariate generating functions are

$$\sum_{m,n\ge 0} C_{m,n}\frac{w^m}{m!}\frac{z^n}{n!} = \frac{e^{w+z}}{(e^w+e^z-e^{w+z})^2}\,; \qquad \sum_{m,n\ge 0} D_{m,n}\frac{w^m}{m!}\frac{z^n}{n!} = \frac{1}{e^w+e^z-e^{w+z}}\,. \tag{15.3}$$

Thus if $H(w,z)$ is the generating function for the D array, the corresponding generating function for the C array turns out to be $G(w,z)H(w,z)$, where $G(w,z)$ is the generating function (1.7) for the B array. Another noteworthy generating function [30] is

$$\sum_{m,n\ge 1} C_{m-1,n-1}\frac{w^m}{m!}\frac{z^n}{n!} \;=\; \ln\frac{1}{e^w+e^z-e^{w+z}}. \tag{15.4}$$

Furthermore, since $e^w+e^z-e^{w+z}=1-(e^w-1)(e^z-1)$, these generating functions lead to the explicit formulas

$$C_{m,n}=\sum_{k\ge 0} k!\,(k+1)!\left\{ {m+1 \atop k+1} \right\}\left\{ {n+1 \atop k+1} \right\}; \qquad D_{m,n}=\sum_{k\ge 0} k!^2\left\{ {m \atop k} \right\}\left\{ {n \atop k} \right\}. \tag{15.5}$$

A bit of fooling around reveals that C's make B's, and D's make C's, in simple ways:

$$B_{m,n} \;=\; C_{m-1,n}+C_{m,n-1}+[m{=}n{=}0]; \tag{15.6}$$

$$C_{m,n} \;=\; D_{m+1,n}+D_{m,n+1}+D_{m,n}. \tag{15.7}$$

We will see that both of these relations have nice combinatorial explanations. (Formula (15.6) is equivalent to Equation (9) in a paper [9] by Arakawa and Kaneko, written in 1999; of course their reasoning at the time was based on analytic number theory, not combinatorics.)

Before proceeding further, let's pause to observe that (15.6) proves a nonobvious property of the pB numbers:

$$\sum_{k=0}^{n}(-1)^k B_{k,n-k} \;=\; [n{=}0]. \tag{15.8}$$

For the sum $(C_{-1,n}+C_{0,n-1})-(C_{0,n-1}+C_{1,n-2})+\cdots+(-1)^{n-1}(C_{n-2,1}+C_{n-1,0})+(-1)^n(C_{n-1,0}+C_{n,-1})$ telescopes to zero when $n>0$. (Relation (15.8) is due to Bényi and Hajnal [2].)

What kinds of combinatorial patterns are enumerated by $C_{m,n}$? We shall see that there are lots and lots of them. For example, $C_{m,n}$ is the number of permutations $p_0p_1\ldots p_{m+n}$ of the $m+n+1$ numbers $[0\,..\,m+n]$ that have the doubly bounded property

$$j-m\le p_j\le j+n \qquad \text{for } 0\le j\le m+n. \tag{15.9}$$

(It just like (8.1), but now there's one more element.)

To prove this, we shall once again evaluate the permanent of a suitable matrix. When $m=4$ and $n=7$ that matrix, with rows and columns numbered from 0 to $m+n$, is

$$Q^+_{m,n} \;=\; \begin{pmatrix} 1&1&1&1&1&1&1&1&0&0&0&0\\ 1&1&1&1&1&1&1&1&1&0&0&0\\ 1&1&1&1&1&1&1&1&1&1&0&0\\ 1&1&1&1&1&1&1&1&1&1&1&0\\ 1&1&1&1&1&1&1&1&1&1&1&1\\ 0&1&1&1&1&1&1&1&1&1&1&1\\ 0&0&1&1&1&1&1&1&1&1&1&1\\ 0&0&0&1&1&1&1&1&1&1&1&1\\ 0&0&0&0&1&1&1&1&1&1&1&1\\ 0&0&0&0&0&1&1&1&1&1&1&1\\ 0&0&0&0&0&0&1&1&1&1&1&1\\ 0&0&0&0&0&0&0&1&1&1&1&1 \end{pmatrix}. \tag{15.10}$$

It's the same as $Q_{m,n}$ in (8.2), but a new column and a new row have been appended at the right and at the bottom: The former S_m has been extended by the all-zero column $O_{m,1}$; the former S_n^T has been extended by the all-zero row $O_{1,n}$; and the former $J_{n,m}$ has become $J_{n+1,m+1}$.

Suppose, as before, we place k rooks into the submatrix $J_{m,n}$ at the upper left. Then, as before, we must place $m-k$ rooks into S_m and $n-k$ rooks into S_n^T. The new twist is that we must now place $k+1$ rooks, not k, into $J_{n+1,m+1}$.

The same argument as before now shows that, given k, the permanent is equal to $k!\,(k+1)!\left\{{m+1\atop k+1}\right\}\left\{{n+1\atop k+1}\right\}$; and that quantity is precisely the term indexed by k in the sum (15.5) for $C_{m,n}$.

Similarly, we can show that $D_{m,n}$ is the number of permutations $p_1\ldots p_{m+n}$ of the $m+n$ numbers $[1\,..\,m+n]$ for which we have

$$j-m< p_j< j+n \qquad \text{for } 1\le j\le m+n. \tag{15.11}$$

The relevant matrix for $m=4$ and $n=7$ now takes the form

$$Q^{++}_{m-1,n-1} \;=\; \begin{pmatrix} 1&1&1&1&1&1&1&0&0&0&0\\ 1&1&1&1&1&1&1&1&0&0&0\\ 1&1&1&1&1&1&1&1&1&0&0\\ 1&1&1&1&1&1&1&1&1&1&0\\ 0&1&1&1&1&1&1&1&1&1&1\\ 0&0&1&1&1&1&1&1&1&1&1\\ 0&0&0&1&1&1&1&1&1&1&1\\ 0&0&0&0&1&1&1&1&1&1&1\\ 0&0&0&0&0&1&1&1&1&1&1\\ 0&0&0&0&0&0&1&1&1&1&1\\ 0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix}, \tag{15.12}$$

where we can see S_3 in rows $(1\,..\,4]$ and columns $[8\,..\,11)$; also S_6^T appears in rows $[5\,..\,11)$ and columns $(1\,..\,7]$. (In general, S_{m-1} will be in rows $(1\,..\,m]$ and columns $[1+n\,..\,m+n)$, whereas S_{n-1}^T will be in rows $[m+1\,..\,m+n)$ and columns $(1\,..\,n]$, assuming that $m,n>0$.) With $m-1-k$ rooks in S_{m-1} and $n-1-k$ rooks in S_{n-1}^T, the permanent comes to $(k+1)!^2\left\{{m\atop k+1}\right\}\left\{{n\atop k+1}\right\}$. This matches (15.5), with k shifted by 1, since the term for $k=0$ in the sum for $D_{m,n}$ is zero when $m+n>0$.

(These enumerations of permutations that don't stray too far from the identity permutation $p_j=j$ were carried out in 1975 by Katalin Vesztergombi [17], who actually proved a considerably stronger three-parameter result: *The number of permutations* $p_1\ldots p_N$ *of* $[1\,..\,N]$ *that satisfy*

$$j-m< p_j< j+n \qquad \textit{for } 1\le j\le N, \tag{15.13}$$

when $0 \le m, n \le N$ *and* $m+n \ge N$*, is* $f(m+n-N, N-m, N-n)$*, where*

$$f(r,s,t) = \sum_{k=0}^{s} (-1)^{k+1} (r+k)!\,(r+k)^t \left\{ {s+1 \atop k+1} \right\}; \quad \sum_{r,s,t} f(r,s,t) \frac{x^r}{r!} \frac{w^s}{s!} \frac{z^t}{t!} = \frac{1}{e^w + w^z - (1+x)e^{w+x}}. \qquad (15.14)$$

Cases $r = 0$, 1, and 2 yield the D, C, and B arrays, respectively. So we should perhaps really be calling all of these arrays "Veztergombi numbers," not pB numbers. The case $r = 0$ had already been resolved by Kaplansky and Riordan in 1946 [28, §8].)

16. Counting significant subclasses. We've defined $\mathcal{V}_{m,n}$ to be the class of all permutations of $[1\,..\,m+n]$ that are characterized by the inequalities $j - m \le p_j \le j + n$, and we know that it has exactly $B_{m,n}$ members. We've just proved, using slightly different notation, that the subclass characterized by the stronger inequalities $j-m < p_j \le j+n$ has exactly $C_{m-1,n}$ permutations; and that the similar subclass characterized by $j-m \le p_j < j+n$ has exactly $C_{m,n-1}$ of them. We've also proved that the intersection of both subclasses, namely those permutations with $j - m < p_j < j + n$, has exactly $D_{m,n}$ members.

Let's denote the first subclass by $\mathcal{V}^{\swarrow}_{m,n}$, and the second subclass by $\mathcal{V}^{\searrow}_{m,n}$, with an arrow pointing to the parameter whose significance became "more strict." It's also convenient to denote the intersection class $\mathcal{V}^{\swarrow}_{m,n} \cap \mathcal{V}^{\searrow}_{m,n}$ by $\mathcal{V}^{\swarrow\searrow}_{m,n}$. Thus,

$$|\mathcal{V}_{m,n}| = B_{m,n}\,; \qquad |\mathcal{V}^{\swarrow}_{m,n}| = C_{m-1,n}\,; \qquad |\mathcal{V}^{\searrow}_{m,n}| = C_{m,n-1}\,; \qquad |\mathcal{V}^{\swarrow\searrow}_{m,n}| = D_{m,n}\,. \qquad (16.1)$$

By analyzing the bijections that we've found between $\mathcal{V}_{m,n}$ and other pB classes, we're now equipped to deduce many similar enumerations of interest. (Most of the results in this section were first presented in a pioneering paper by Beáta Bényi and Péter Hajnal [25].)

The easiest bijection with $\mathcal{V}_{m,n}$ is rule (9.3), which connects it to $\mathcal{E}_{m,n}$. According to (9.3), $\mathcal{E}^{\swarrow}_{m,n}$ is the set of permutations with $q_j > j$ for $1 \le j \le m$ and $q_j \le j$ for $m < j \le m+n$; thus it consists of the permutations $q_1 \ldots q_{m+n}$ whose excedance set is precisely $[1\,..\,m]$. (The number of such permutations, namely $|\mathcal{E}^{\swarrow}_{m,n}| = C_{m-1,n}$, was first found by Ehrenborg and Steingrímsson in 2000 [29]; see also [16].)

Similarly, $\mathcal{E}^{\searrow}_{m,n}$ contains the permutations with $q_j \ge j$ for $1 \le j \le m$ and $q_j < j$ for $m < j \le m+n$; their *weak* excedance set is precisely $[1\,..\,m]$, and there are exactly $C_{m,n-1}$ of them.

Finally, $\mathcal{E}^{\swarrow\searrow}_{m,n}$ consists of the $D_{m,n}$ permutations of $\mathcal{E}_{m,n}$ that have no fixed points.

What about parades? The set $\mathcal{P}^{\swarrow}_{m,n}$ that corresponds to $\mathcal{V}^{\swarrow}_{m,n}$ in the bijection of §8 is a bit peculiar: It comes from permutations that have no rooks on the diagonal of S_m in $Q_{m,n}$ (see (8.2)). According to Bender's bijection, those are the parades that prevent that the restricted growth string for the girls from putting "adjacent" girls g_i and g_{i+1} into the same block of the set partition, for $0 \le i < m$. Paradewise, it means that (i) the parade doesn't begin with g_1; and (ii) no girls g_i and g_{i+1} who are adjacent in age are adjacent in the parade. There are $|\mathcal{P}^{\swarrow}_{m,n}| = C_{m-1,n}$ such parades.

Similarly, $\mathcal{P}^{\searrow}_{m,n}$ consists of the parades that don't end with b_1 or have adjacent-in-age boys next to each other in the parade; and $\mathcal{P}^{\swarrow\searrow}_{m,n}$ is the intersection. In particular,

$$\mathcal{P}^{\swarrow\searrow}_{2,2} = \{g_2b_1g_1b_2,\ b_1g_1b_2g_2,\ b_1g_2b_2g_1,\ b_2g_1b_1g_2,\ b_2g_2b_1g_1\}. \qquad (16.2)$$

Of course there are more interesting subclasses of parades. For instance, how many parades end with a girl? These are the parades that don't end with a boy; and the bijection between $\mathcal{P}$ and $\mathcal{V}$ in §8 will put a boy at the end if and only a rook is placed into one of the first n columns of row $m+1$ in $Q_{m,n}$.

Let's take a closer look at that bijection. If $p_1 \ldots p_{m+n}$ is the permutation of rooks that corresponds to a parade Π, as (8.5) corresponds to (3.9), let $q_1 \ldots q_{m+n}$ be the inverse permutation. For example, we have

$$q_1 \ldots q_{11} \;=\; 1\;7\;2\;8\;5\;3\;6\;4\;9\;10\;11 \qquad (16.3)$$

in (8.4) and (8.5); q_j is the location of the rook in column j.

According to Bender's bijection, girl g_i will be first in Π if and only if $q_{n+1} = i$; that's the condition for $a_1 \ne 0, \ldots, a_{i-1} \ne 0$, $a_i = 0$ in the restricted growth string for the girls. Similarly, boy b_j will be in the block at the end of the parade if and only if $p_{m+1} = j$, if and only if b_j is first in the left-right *reflected* parade Π^R. (In general, Π corresponds to $p_1 \ldots p_{m+n}$ in $\mathcal{V}_{m,n}$ if and only if Π^R corresponds to $q_1 \ldots q_{m+n}$ in $\mathcal{V}_{n,m}$.)

Let $\mathcal{V}^{\nearrow}_{m,n}$ be the subclass of $\mathcal{V}_{m,n}$ whose permutations satisfy $p_{m+1} \le n$. It corresponds to $\mathcal{P}^{\nearrow}_{m,n}$, the subclass of $\mathcal{P}_{m,n}$ whose parades end with a boy. And $|\mathcal{V}^{\nearrow}_{m,n}|$ is the permanent of the matrix obtained by zeroing out the m entries in the top row of $J_{n,m}$ within $Q_{m,n}$ (see (8.2)).

Now here's a neat trick: If we move that modified row of $Q_{m,n}$ up to the top, we get the matrix $Q^{+}_{m,n-1}$ that's illustrated in (15.10)! Consequently $|\mathcal{V}^{\nearrow}_{m,n}| = |\mathcal{P}^{\nearrow}_{m,n}| = C_{m,n-1}$.

Similarly, the subclass $\mathcal{V}^{\nwarrow}_{m,n}$ whose permutations satisfy $q_{n+1} \le m$ has $C_{m-1,n}$ elements. (Zero out the leftmost column of $J_{n,m}$.) It corresponds to $\mathcal{P}^{\nwarrow}_{m,n}$, the parades that begin with a girl.

The reflection of a parade that begins with a girl always ends with a girl, and vice versa. As a consequence, we've got a nice combinatorial explanation for identity (15.6): The $B_{m,n}$ parades consisting of m girls and n boys divide into exactly $C_{m-1,n}$ that start with a girl and $C_{m,n-1}$ that start with a boy, except of course when $m = n = 0$.

The same approach can be used to deduce the number of parades that begin with a girl and end with a boy. It's tempting to call that subclass $\mathcal{P}^{\nwarrow\!\!\!\nearrow}_{m,n}$, because it's the intersection of $\mathcal{P}^{\nwarrow}_{m,n}$ and $\mathcal{P}^{\nearrow}_{m,n}$. However, we have to be sneaky in order to determine its size. The value of $|\mathcal{V}^{\nwarrow}_{m,n} \cap \mathcal{V}^{\nearrow}_{m,n}|$ is the permanent of a rather strange matrix that cannot be converted to a matrix like (15.10) or (15.12) by row and column permutations, because its nth row and its mth column are both all 1s.

Instead, let's say that $\mathcal{P}^{\nwarrow\!\!\!\nearrow}_{m,n}$ consists of the parades of $\mathcal{P}_{m,n}$ that begin with a boy and end with a girl. They're the reflections of the girl-first-boy-last ones, so their number is the same. (It's *not* the intersection $\mathcal{P}^{\nwarrow}_{m,n} \cap \mathcal{P}^{\nearrow}_{m,n}$, but it's bijectively equivalent to that intersection.)

And the corresponding permutations $\mathcal{V}^{\nwarrow\!\!\!\nearrow}_{m,n}$ are readily enumerated, because $|\mathcal{V}^{\nwarrow\!\!\!\nearrow}_{m,n}|$ is the permanent of the matrix obtained from $Q_{m,n}$ by zeroing out the left column of S_m and the top row of S_n^T. When column $n+1$ of that matrix is moved to the far right, and row $m+1$ is moved to the very bottom, we get $Q^{++}_{m-1,n-1}$ (see (15.12)). Hence the permanent is $D_{m,n}$.

Good! We've just proved a set of formulas that's dual to (16.1):

$$|\mathcal{V}_{m,n}| = B_{m,n}\,; \qquad |\mathcal{V}^{\nwarrow}_{m,n}| = C_{m-1,n}\,; \qquad |\mathcal{V}^{\nearrow}_{m,n}| = C_{m,n-1}\,; \qquad |\mathcal{V}^{\nwarrow\!\!\!\nearrow}_{m,n}| = D_{m,n}\,. \tag{16.4}$$

Also, for the record,

$$\mathcal{P}^{\nwarrow\!\!\!\nearrow}_{2,2} = \{b_1b_2g_1g_2,\ b_1g_1b_2g_2,\ b_1g_2b_2g_1,\ b_2g_1b_1g_2,\ b_2g_2b_1g_1\}. \tag{16.5}$$

What about parades that both begin and end with a girl? That's fun to work out, but it's a bit more subtle. We shall subdivide such parades into two further subclasses, $\mathcal{P}^{\dagger}_{m,n}$ where g_m does *not* begin the parade, and $\mathcal{P}^{\ddagger}_{m,n}$, where she *does*. (Yes, we're running out of symbols.)

The relevant 0–1 matrix for rook placement in the first case (†) is obtained from $Q_{m,n}$ in (8.2) by zeroing out the top row of S_n^T, the left column of $J_{n,m}$, and the 1 at the bottom left corner of S_m. Moving row $m+1$ down to the bottom, then moving column $n+1$ to the left, gives the matrix $Q^{++}_{m-1,n+1}$ (see (15.12)). So $|\mathcal{V}^{\dagger}_{m,n}| = |\mathcal{P}^{\dagger}_{m,n}| = D_{m-1,n+1}$.

The relevant matrix in the second case (‡) is obtained by zeroing out the top row of S_n^T, then moving row $m+1$ to the bottom, and deleting row m and deleting column $n+1$. The result is $Q^{++}_{m-1,n}$, whose permanent is $D_{m-1,n}$. (Another way to handle this case, when $n > 0$, is to realize that g_m can begin a parade only if she is immediately followed by a boy. And if the parade that follows her ends with a girl, it must be the reflection of one of the $D_{m-1,n}$ parades in $\mathcal{P}^{\nwarrow\!\!\!\nearrow}_{m-1,n}$. On the other hand, if $n = 0$, a parade can begin with g_m if and only if $m = 1$.)

In summary, we've shown that

$$|\mathcal{P}^{\dagger}_{m,n}| = D_{m-1,n+1} \qquad \text{and} \qquad |\mathcal{P}^{\ddagger}_{m,n}| = D_{m-1,n+1}. \tag{16.6}$$

The number of parades in $\mathcal{P}_{m,n}$ that begin and end with a girl is the sum of these two. Hence the total number of parades that begin with a girl, namely $C_{m-1,n}$, is that sum plus $|\mathcal{P}^{\nwarrow\!\!\!\nearrow}_{m,n}|$, namely $D_{m-1,n+1}+D_{m-1,n}+D_{m,n}$. This amounts to a combinatorial proof of identity (15.7), with m shifted by 1.

The number of parades in $\mathcal{P}_{m,n}$ that begin and end with a boy is obtained by interchanging m and n; so it is $D_{m+1,n-1} + D_{m,n-1}$.

Exercise 16.1. What are the elements of (a) $\mathcal{E}^{\nwarrow}_{m,n}$, (b) $\mathcal{E}^{\nearrow}_{m,n}$, (c) $\mathcal{E}^{\nwarrow\!\!\!\nearrow}_{m,n}$, (d) $\mathcal{E}^{\dagger}_{m,n}$, (e) $\mathcal{E}^{\ddagger}_{m,n}$?

17. The other pB subclasses.

Open problems. TO BE WRITTEN

how many equivalence classes of parades under automorphisms?

all 120 pairs of 2x2 matrices M: (M,M') free has 14 for m=n=2; how many pB?

another way to represent parades via Young tableaux?

pad with zeroes so that $m \times n$ becomes $\infty \times \infty$?

enumerate (weakly) Gamma-free; Spinrad has asymptotics

enumerate maxclosed in three or more dimensions

exploit identity $c_{m,n} = \sum_{k,l} \binom{m}{k}\binom{n}{l} b_{k,l} d_{m-k,n-l}$

prepare expository videos about the theory of parades, oriented to high school students (great opportunities for animation, costumes, music!)

Appendix. The fundamental recurrence $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ leads to an easy way to find the rth sample $1 \le x_0 < \cdots < x_{k-1} \le n$ from the interval $[1\,..\,n]$, given $0 \le k \le n$ and $0 \le r < \binom{n}{k}$:

$$\text{“While } k > 0, \text{ test if } r < \tbinom{n-1}{k}\text{; if not, set } r \leftarrow r - \tbinom{n-1}{k}, \; k \leftarrow k-1, \text{ and } x_k \leftarrow n\text{; then in either case set } n \leftarrow n-1.\text{”} \tag{A.1}$$

The samples are obtained in colexicographic order: $b_1 \ldots b_{k-1} b_k$, $b_1 \ldots b_{k-2} b_{k-1} b_{k+1}$, $b_1 \ldots b_{k-2} b_k b_{k+1}$, $\ldots$, $b_{n-k+1} b_{n-k+2} \ldots b_n$. Conversely,

$$\operatorname{rank}(x_0 x_1 \ldots x_{k-1}) = \binom{x_0 - 1}{1} + \binom{x_1 - 1}{2} + \cdots + \binom{x_{k-1} - 1}{k}. \tag{A.2}$$

(Theorem 7.2.1.3L in [23] traces this formula to Ernesto Pascal in 1887.)

Here is an algorithm that can be used to transform (5.3) into (5.4). It finds the parade Π of $m+1$ girls and n boys that extends a given parade Π' of m and n' boys by a given subset $\{b_{x_0}, \ldots, b_{x_{k-1}}\}$. Here $n' = n$ if $k = 0$, otherwise $n' = n + 1 - k$. The boy represented by x_{k-1} is called “Max” in the comments below. His former place in the parade will be replaced by g_{m+1} followed by the subset that he leads. The algorithm is conceptually simple; but, as usual, God (or the devil) is in the details.

The parades are represented by two digits strings, $s_0 \ldots s_{m+1}$ for the girls and $t_0 \ldots t_n$ for the boys, as discussed in §2. For example, the digit strings for the parade $\Pi_{4.7,999999}$ in (5.4) are $s_0 \ldots s_4 = 02331$ and $t_0 \ldots t_7 = 02110302$. (Initially only $s_0 \ldots s_m$, $t_0 \ldots t_{n'}$, and $x_0 \ldots x_{k-1}$ are given. All operations take place within those arrays, without needing any auxiliary memory.)

We assume that partition Π' has order d. In other words, both of the ordered set partitions initially have $d+1$ nonempty blocks. Block 0 for the girls corresponds to set $S_0 \cup \{g_0\}$, but block 0 for the boys corresponds to set $T_{d+1} \cup \{b_0\}$, as in §2 and §3 above. The algorithm increases d by 1 if the output parade Π has higher order than Π'.

X1. [Empty case?] If $k = 0$, go to step X10.

X2. [Is Max alone?] Set $\mu \leftarrow x_{k-1} - (k-1)$, $p \leftarrow t_\mu$, and $q \leftarrow -1$. Then for $1 \le j \le n'$, set $q \leftarrow q + 1$ if $t_j = p$. Go to step X4 if $q = 0$.

X3. [Split block p.] Set $q \leftarrow 1$ and $d \leftarrow d + 1$. If $p = 0$, set $t_j \leftarrow d$ for $1 \le j \le n'$.

X4. [Begin the loop.] Set $i \leftarrow n'$, $j \leftarrow n$, and $l \leftarrow k - 2$.

X5. [Loop done?] (At this point $i \le j$.) Go to X8 if $i = 0$.

X6. [Bypass the subset, except Max.] While $l \ge 0$ and $j = x_l$, set $l \leftarrow l - 1$, $j \leftarrow j - 1$, and repeat this step.

X7. [Update t_j.] Set $p' \leftarrow t_i$. If $q = 1$ and $p' > p > 0$, set $t_j \leftarrow p' + 1$; otherwise set $t_j \leftarrow p'$. Then set $i \leftarrow i - 1$, $j \leftarrow j - 1$, and return to X5. (Boy b_j has been renamed.)

X8. [Update the subset.] For $0 \le l < k$, set $t_{x_l} \leftarrow 0$ if $p = 0$, otherwise set $t_{x_l} \leftarrow p + q$.

X9. [Update the girls.] If $p = 0$, set $s_{m+1} \leftarrow d$. Otherwise, if $q = 0$, set $s_{m+1} \leftarrow p - 1$. Otherwise, set $s_{m+1} \leftarrow p$, and also set $s_j \leftarrow s_j + 1$ for all $j \in [1\,..\,m]$ with $s_j \ge p$. Terminate the algorithm.

X10. [Extend by $\emptyset$.] (We will change Π' by simply appending g_{m+1} at the end.) If $t_1, \ldots, t_n$ are all nonzero, go to X12. (Otherwise there was at least one boy at the end.)

X11. [Increase the order.] Set $d \leftarrow d+1$. Then set $t_j \leftarrow d$ for all $j \in [1\,..\,n]$ with $t_j = 0$.

X12. [Append g_{m+1}.] Set $s_{m+1} \leftarrow d$ and terminate the algorithm. ▮

An experimental implementation of this algorithm, together with another program that unranks parades according to the nonrecursive scheme of §3, can be found online [27]. That website also contains programs for the corresponding ranking algorithms.

Acknowledgments. The author heartily thanks Ira Gessel for getting him started on this topic. In particular, Ira suggested formula (1.8), the bijection for acyclic orientations in §7, and the bijection for max-closed relations in §14.

Answers to the exercises.

8.1. Place rooks into $Q_{16,20}$, using σ, Σ, τ, and T from (2.4):

$$\left(\begin{array}{cccccccccccccccccccccccccccccccccccc}
1&1&1&1&\text{♜}&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&\text{♜}&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&\text{♜}&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&\text{♜}&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&\text{♜}&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&\text{♜}&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0\\
1&1&\text{♜}&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&\text{♜}&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&\text{♜}&1&1&1&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&\text{♜}&1&1&1&1&1&1&1&1&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&\text{♜}&1&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&\text{♜}&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&\text{♜}&1&1&1&1&1&1&0&0&0\\
1&1&1&1&1&1&1&1&1&1&\text{♜}&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&\text{♜}&1&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&\text{♜}&1&1&1&1&1\\
\text{♜}&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&1&1&1&1&1&1&1&1&1&\text{♜}&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&1&1&1&1&\text{♜}&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&\text{♜}&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&\text{♜}&1&1&1&1&1&1&1\\
0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&\text{♜}&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&1&1&1&1&1&1&\text{♜}&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&1&1&1&1&1&1&\text{♜}&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&\text{♜}&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&\text{♜}&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&\text{♜}&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&\text{♜}&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&\text{♜}&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&\text{♜}&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&\text{♜}&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&\text{♜}&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&\text{♜}\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&\text{♜}&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&\text{♜}&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&\text{♜}&1
\end{array}\right)$$

So $p_1 \ldots p_{36} =$ 5 2 6 23 8 17 3 10 26 22 30 15 27 11 34 31 1 12 7 4 20 16 13 14 0 18 33 24 19 21 32 28 36 25 20 35. (That permutation faithfully encodes the values $\lfloor 10^{15}\pi \rfloor$ and $\lfloor 10^{19}\gamma \rfloor$, because of (2.3)!)

11.1. Begin with the parade of $\mathcal{P}_{0,4}$. Extend it first by $\{b_1\}$ $(b_2 b_3 b_4 g_1 b_1)$; then by $\{b_1 b_3\}$ $(b_4 b_5 g_2 b_1 b_3 g_1 b_2)$; then by $\{b_1 b_5\}$ $(b_6 g_3 b_1 b_5 g_2 b_2 b_4 g_1 b_3$; finally by $\{b_1 b_7\}$ $(g_4 b_1 b_7 g_3 b_2 b_6 g_2 b_3 b_5 g_1 b_4)$. [The recursive rank is 999161.]

11.2. Since Π has type $\{b_{10}\}$ we know that the first row puts 1 in column 10. Then Π', of type $\{b_1 b_{12} b_{19}\}$, gives us 1s in columns 1, 12, 19, and zeros below the first two of those 1s. Then Π'', of type $\{b_{15}\}$, puts a 1 into column 17, since b_{15} was originally named b_{17}. Here's the glorious final result:

$$\begin{pmatrix}
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&1&0&1\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&1&1&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0\\
0&1&0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}$$

The author's program RANK-PARADE2 (see [27]) displays types with original boys' names as well as their current names, thereby making it easy to "read off" the rows of this matrix when fed the parade (2.2).

11.3. The solution to the recurrence $x_{1,n} = n$, $x_{m+1,n} = \max\{1 + x_{m,n-1}, 2 + x_{m,n-2}, \ldots, n + x_{m,1}\}$ is $x_{m,n} = m + n - 1$. (This recurrence is derived from the construction of $\Gamma_{m,n,r}$.)

13.1. $\begin{pmatrix} 1&1&1&1&1&1&1\\ 0&1&1&1&1&1&0\\ 0&0&1&1&1&0&0\\ 0&0&0&1&0&0&0 \end{pmatrix}$.

13.2. The text's bijection from Γ to Λ should be applied to the matrix Γ of answer 11.2 from bottom to top. That is, we start with the bottom row of Γ, getting our initial version of the bottom row of Λ; then we use the next-to-last and last rows of Γ to get an initial version of the bottom two rows of Λ, namely $\binom{00000000010000010001}{00000000010000000001}$; and so on. The result is

$$\Lambda = \begin{pmatrix}
1&1&1&1&0&0&1&1&1&1&1&1&0&1&0&0&1&1&1&1\\
1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0\\
1&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&1&0&1&0\\
1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0\\
1&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&0\\
1&0&1&1&0&0&0&1&1&0&1&1&0&0&0&0&1&1&1&1\\
1&1&1&1&0&0&1&1&1&1&1&1&0&1&0&0&1&1&1&1\\
1&0&1&1&0&0&0&1&1&0&1&1&0&0&0&0&1&1&1&1\\
1&0&1&1&0&0&0&0&0&0&1&1&0&0&0&0&1&0&1&0\\
1&1&1&1&0&0&1&1&1&1&1&1&0&1&1&0&1&1&1&1\\
1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0\\
1&0&1&1&0&0&0&1&1&0&1&1&0&0&0&0&1&1&1&1\\
1&1&1&1&1&0&1&1&1&1&1&1&0&1&1&1&1&1&1&1\\
1&1&1&1&0&0&1&1&1&0&1&1&0&1&0&0&1&1&1&1\\
1&1&1&1&1&0&1&1&1&1&1&1&0&1&1&1&1&1&1&1\\
1&1&1&1&0&0&1&1&1&1&1&1&0&1&0&0&1&1&1&1
\end{pmatrix}.$$

[The row sums $(r_1, \ldots, r_{16})$ are (15, 3, 5, 3, 4, 11, 15, 11, 7, 16, 3, 11, 18, 14, 18, 15) and the column sums $(c_1, \ldots, c_{20})$ are (16, 7, 11, 11, 2, 0, 7, 10, 10, 6, 13, 16, 0, 7, 3, 2, 12, 10, 16 10). Of course they determine Λ uniquely, since Λ is lonesome.]

Since exercise 11.2 obtained the original matrix Γ from the parade (2.2), we might expect Λ to be precisely the lonesome matrix that corresponds to (2.2) according to the bijection of §10. In fact, however, (2.2) corresponds to the top-to-bottom *reflection* of Λ, because the top row of Λ corresponds to the orientation of arcs from g_{16}, not g_1. (The parade that does correspond to Λ is obtained from (2.2) by swapping $g_j \leftrightarrow g_{17-j}$.)

13.3. Let $p_0 = \infty$ and $p_{m+1} = 0$. If $p_{i-1} > p_i = \cdots = p_j > p_{j+1}$ and $i \le j$, put 1 into columns $p_{j+1}, \ldots, p_i$ of row i, and put 1 into column p_i (only), in rows $i+1, \ldots, j$. (Notice that going from Λ to Γ in the inverse bijection requires us to zero out columns $j_1, \ldots, j_{t-1}$ of Γ'.)

13.4. If Λ' is nonzero and the top row of Γ is nonzero, we must have $t = 1$. Hence the matrices with $\Lambda = \Gamma$ are of three kinds: (i) at most one 1; (ii) more than one 1, all in a single row or a single column; (iii) more than one 1 in row i and more than one 1 in column j, with a 1 in cell (i,j) but no 1s in column j below row i. The number of possibilities is therefore the sum of (i) $mn+1$; (ii) $m(2^n - n - 1) + (2^m - m - 1)n$; (iii) $\sum_{i=2}^{m}(2^{i-1}-1)\sum_{k=2}^{n}\binom{n}{k}k = (2^m - m - 1)(2^{n-1}-1)n$.

13.5. (a) True. (b) True. (c) True. It's a marvelously simple bijection.

16.1. In addition to the basic constraints $p_j \ge j$ for $1 \le j \le m$ and $p_j \le j$ for $m < j \le m+n$, impose further constraints as follows:

(a) $p_j \ne m$ for $j > m$.
(b) $p_{m+n} > m$.
(c) $p_j \ne m$ for $j \le m$, and $p_{m+n} \le m$.
(d) $p_1 \ne m$; $p_j \ne m$ for $j > m$; $p_{m+n} < m$.
(e) $p_1 = m$; $p_{m+n} < m$.

[So we know how to count these subclasses. Too bad they aren't likely to arise in any practical problems.]

References. [RENUMBER AND REORDER THESE!]

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