Signed Skeletons

(Don Knuth, Stanford Computer Science Department)
(24 April 2020)
(dedicated to the memory of John H. Conway)

While studying 1970ish papers on computer vision [1][2], I stumbled onto some fascinating elementary questions about Euclidean 3D geometry that are making my head whirl. I've always been poor at geometric intuition and visualization; but this problem has demonstrated my deficiencies more dramatically than anything else I've ever encountered. On the other hand, I believe that there are many people with the ability to extend the meagre results I was able to achieve, and who might well be inspired to solve some of the open(?) problems at the end of this note.

Consider the set of all three-dimensional polyhedra for which every vertex belongs to exactly three faces. I'll call such an object a 3VP, which abbreviates "three-valent polyhedral object."

The simplest 3VP is a tetrahedron. Many other examples also immediately come to mind, such as a cube, or any polygonal prism. Dodecahedra, sure. But an octahedron does *not* qualify, because each vertex of an octahedron is four-valent. The Great Pyramid of Giza does not qualify, because four faces meet at its apex.

Let me try to make the ground rules more clear. Every face of a 3VP is flat; it's a polygon in some plane, pierced by zero or more polygonal holes. A 3VP might contain interior three-dimensional holes that are totally invisible, not actually part of it yet not connected to the outside without passing through the object. There might also be wormholes, connected to the outside but invisible unless you could sneak a camera inside somehow.

Each 3VP must be a compact set (closed and bounded). Hence its faces are also compact. Moreover, suppose X is a 3VP and F is one of its faces; then X lies "on one side" of F. This means, more precisely, that there's a unique unit vector u with the following three properties: (i) All points $f \in F$ satisfy $u \cdot f = c$, for some constant c; (ii) If f is in the interior of F, then $f + \epsilon u \in X$ for all sufficiently small $\epsilon > 0$; and (iii) If f is in the interior of F, then $f - \epsilon u \notin X$ for all sufficiently small $\epsilon > 0$.

Just in case you're wondering, I insist that a 3VP must have only finitely many vertices. (That's probably a consequence of being compact; I confess that I'm not sure. As a computer scientist, I stopped being able to understand infinity a long time ago.)

Please notice that I do *not* require a 3VP to be connected: The union of two pointwise disjoint 3VP is also a 3VP. In particular, two or more of the connected components can be interlinked like rings, or entangled in more bizarre ways. Furthermore, by adding just four new vertices, we can obtain a "complement" of any 3VP X by adding the faces of X to $T \setminus X$, when T is any large tetrahedron whose interior encloses X. Thus we can have, for example, objects inside of the holes that are inside of other objects. Unions and intersections of other kinds are also permissible; but I don't want to pursue that topic here.

A 3VP must be "physically realizable," with actual (x, y, z) coordinates specified for every vertex, and with the sequence of vertices for each polygonal boundary specified for every face. In particular, those polygons must not be self-intersecting; the polygons that define interior holes must really be interior; etc. Every edge must be a segment of exactly two polygons, belonging to two adjacent faces.

Two 3VPs are "equivalent" if we can continuously morph one into the other while preserving 3VPness. (I suppose this is what my prof at Caltech would have called "homotopically equivalent" when I took his topology course in 1961.)

By definition, each vertex of a 3VP belongs to exactly three edges, each of which is either *convex* or *concave*. The vertex is said to be of type 0, 1, 2, or 3 according as it has 0, 1, 2, or 3 concave edges. This property is preserved under equivalence, because a transition between convex and concave can happen only when adjacent faces are coplanar — which they cannot be. (An edge is always a line segment in the intersection of two planes.)

*   *   *

Excuse me for taking so many words to describe what I mean by the intuitively obvious concept of a 3VP. Now let me come at last to the main point: Let X be a 3VP. The *skeleton* of X is the (cubic) graph that is obtained when we consider only the vertices and edges of X. And we can mark each of those edges as either convex or concave. Let's say that a graph with marked edges is "signed"; and let's say that the *signature* of X is its signed skeleton. This signed skeleton is preserved by homotopic equivalence.

I naturally asked myself, "Given a signed cubic graph G, is it the signature of at least one 3VP?" In such a case I call G "realizable".

I started to construct as many interesting 3VPs as I could, and I kept a catalog of their signatures. And the inverse question, stated in the preceding paragraph, turned out to be extremely fascinating; indeed I could hardly think about anything else for several days. On Thursday I took note of graphs that I thought were surely unrealizable; but on Friday or Saturday, I surprised myself by finding appropriate (x, y, z) coordinates for several of those cases.

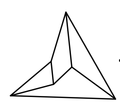
So I took out my copy of Read and Wilson's *Atlas of Graphs* [3], hoping to categorize all realizable cases with up to 10 vertices — because those graphs can all be listed on a single page. (The list is on page 127, if you have access to the book.) Read and Wilson named those graphs C1 through C27, where C1 is K_4, C2 is $K_3 \,\square\, K_2$, and C3 is $K_{3,3}$; C27 is the Petersen graph.

At the moment I've decided to stop at C8, however — the last 8-vertex graph in their list — because I can easily foresee dozens and dozens of realizations of the nineteen 10-vertex graphs, and I really am supposed to be writing Section 7.2.2.3 of *The Art of Computer Programming*.

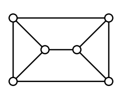
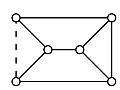
The smallest cases. In general, a cubic graph has $2k$ vertices and $3k$ edges, for some $k > 1$.

Graph C1, the only cubic graph on four vertices, is obviously realized by a tetrahedron. And I believe it obviously cannot be realized with concave edges: It must be the convex hull of its four vertices, in any realization.

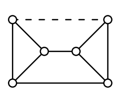
Graph C2, which consists of two copies of the triangle K_3 joined together with three edges, is obviously realized by a triangular prism. But C2 can also be realized with one concave edge, by making a "crease" in one face of a tetrahedron:

 .

To represent a signed skeleton "graphically," I'll use solid lines for convex edges and dashed lines for concave edges (thinking of "mountain folds" and "valley folds" in origami). Thus, the two realizations of C2 mentioned already have the respective signatures

 and  .

There's also a third potential signature for C2, namely

 .

I do *not* believe that it is realizable, but at the moment I have no rigorous proof of impossibility.

The other 6-vertex cubic graph is C3, which Dudeney made famous as the "water, gas, and electricity problem"[4]: To connect three houses to three utilities without any crossing of pipes. It's the smallest *nonplanar* cubic graph, $K_{3,3}$.

Planarity is not really a severe restriction on realizability. We can take an arbitrary finite graph and build it in 3D out of wires, then fashion some kind of tubes around the wires so as to have a 3VP for which the original graph is a minor.

However, such a construction involves a nontrivial number of auxiliary vertices, and I'm quite certain that six vertices isn't enough to realize a nonplanar skeleton. Thus I'm willing to bet that C3 is unrealizable. Still, as above, I don't know of any rigorous way to prove such an assertion.

Systematic constructions. I thought of four main ways to construct small realizations. First, given any realization of a signed graph G, I can choose a vertex v and its three neighbors $\{a, b, c\}$. I replace v by a triangle $\{a', b', c'\}$, with edges a' to a, b' to b, c' to c. This can always be done in at least two ways, one by "lopping off" a wedge and the other by "gluing on" a wedge; the results differ in the number of concave edges formed.

(The "lopping off" technique is essentially how Steinitz [5] proved by induction that every 3-connected cubic graph can be realized as a convex polyhedron. His theorem tells us that such graphs always have a purely convex signature.)

A second way is to choose an edge uv, with $\{a, b\}$ the other neighbors of u and $\{c, d\}$ the other neighbors of v; here $\{a, u, v, c\}$ are on one face, $\{b, u, v, d\}$ are on another; possibly $a = c$ or $b = d$. Replace that edge by a 4-cycle u' to v' to v to u to u'; let b and d now be neighbors of u' and v' instead of u and v. Again there are at least two possibilities, one by shearing and one by gluing.

A third way, which I haven't explored fully, is to make a "crease" in one of the (outermost?) faces, bending the half-face on one side of the crease inward or outward while adjusting its vertices to where the former lines intersect the newly tilted plane.

Each of those methods increases the number of vertices by 2.

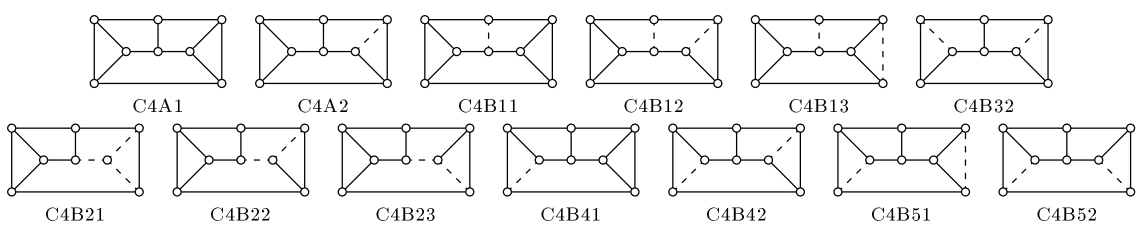
A fourth way is to take two disjoint 3VPs and to glue an outermost face of one of them onto the middle of some face on the other one (shrinking it first if needed). Or, similarly, to punch out its inverse . . . I mean, to intersect the host with its complement.

I could also, for instance, take a large tetrahedron and a small tetrahedron, and use the small one to drill a hole through the large one. That gives a (disconnected) signed graph with 10 vertices: 4 for the original big tetrahedron, and 6 for the vertices in a hollowed-out triangular prism. There are three concave edges inside of this toroid, twelve convex edges outside.

Sometimes, however, I just eyeballed the given graph and tried to imagine building it somehow.

A complete(?) catalog for eight vertices. Exactly six cubic graphs have 8 vertices (and hence 12 edges). Two of them, called C5 and C7 by Read and Wilson [3], are nonplanar; again I believe that they're unrealizable.

C4 proved to be the most amazing graph that I considered. You can get it by lopping a corner off of a triangular prism; but there are other ways to create it while introducing concave edges. For several days I kept finding new possibilities, until finally coming up with the following list of thirteen nonisomorphic signatures:



I believe that this list is complete, although I won't be unhappy to be proved wrong.

Several of these have realizations that are not homotopically equivalent. For example, there are chiral (left-handed versus right-handed) variations. I haven't had time to explore anything more than to find at least one realization for each signature shown above.

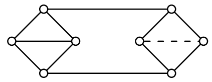
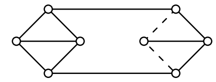
But since I'm so poor at visualization, I *did* take time to create an interactive website, on which you can visualize all thirteen realizations by rotating them to your heart's content: Just go to your laptop, tablet, or smartphone and point a modern browser to the URL

`http://cs.stanford.edu/~knuth/Polyhedra-DEK.html`

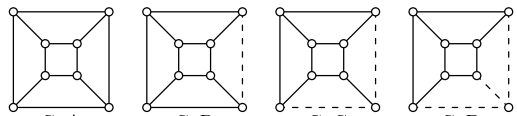
and use your fingers (or a mouse)! All thirteen realizations are available for viewing, identified by the names C4A1 through C4B52 shown above.

The next graph in Read and Wilson's atlas, C6, is planar but not 3-connected. Steinitz's Theorem in [5] not only asserted that 3-connected cubic graphs are convexly realizable, he also proved the converse: The skeleton of every convex 3VP is 3-connected. Therefore C6 has no all-convex signature.

But C6 *does* have two (and I believe only two) realizable signatures:

 and .

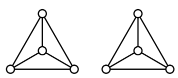
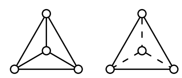
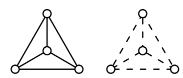
Graph C8, $K_2 \,\square\, K_2 \,\square\, K_2$, is the well-known 3-cube, with 12 convex edges. But it also has three other signatures:



C8A C8B C8C C8D .

(See the website; the realizations found there were obtained systematically from the two realizations of C2. Again, I don't know of any other signatures for which C8 is realizable as a 3VP.)

Finally, the remaining eight-vertex cubic graph, $K_4 \oplus K_4$, is disconnected. We can combine one tetrahedron with another one, possibly smaller, in various ways. I think there are three realizable signatures, namely

, , ,

but with two nonhomotopic variations of the middle one (additive versus subtractive).

Some open(?) problems. So that's basically all I know about 3VPs, except that I'm more than 99% certain that a lot more fascinating properties remain to be discovered (if they haven't already). Let me close this note by listing several possibly-open problems for which I'd dearly like to know the answers:

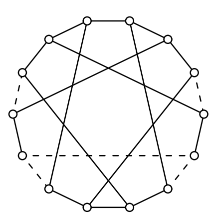
1. Is there an algorithm to decide whether or not a given signed cubic graph is realizable? (In particular, such an algorithm could be used to discover if I've omitted any cases with at most eight vertices.)

2. Is there an algorithm to construct all of the equivalence classes (under homotopy) of the realizations of a given signed cubic graph? (Who knows how many of my thirteen signed graphs for C4 will split into two or more equivalent classes?)

3. If there exists a realization of some signed G, does there always also exist a realization in which all coordinates (x, y, z) are integers?

4. If there exists a realization of some signed G, does there always also exist a realization in which all angles between faces are multiples of 30 degrees? (This seems to have been true in all the cases I looked at, including several with 10 or more vertices, but I'm not really very certain of this.)

5. What's the smallest n for which there's a realizable nonplanar graph with n vertices? I believe that the answer is $n = 14$. The famous Szilassi polyhedron [6], whose seven faces each touch the other six, has the nonplanar signature

,

which is a signed version of graph C621 in [3] — also known as the Heawood graph. (You can view it and play with it on the website mentioned above.) It may well be a counterexample to the conjecture in problem 4. Does the Heawood graph have any other signature?

Epilogue. In closing, I can't help mentioning that I've been seeing examples or near-examples of 3VP ever since I began this investigation. The other day, while getting some exercise despite social-distancing, I noticed a striking statue in front of the SAP Labs in Stanford's Research Park [7]. I was sorely tempted to model it in my computer, with slight simplifications, so that I could add it to the collection of examples in my interactive website.

I've managed to resist that temptation, because other duty calls. But I think a sculpture of that kind might be even more appealing if it were *explicitly* designed to satisfy the 3VP constraints, while retaining the grace and vivacity of the idealized dancers. What do you think?

References.

[1] M. B. Clowes, "On seeing things," *Artificial Intelligence* **2** (1971), 79–116.

[2] D. A. Huffman, "Impossible objects as nonsense sentences," *Machine Intelligence* **6** (1971), 295–323.

[3] Ronald C. Read and Robin J. Wilson, *An Atlas of Graphs* (Oxford: Clarendon Press, 1998), xi+454 pages.

[4] Henry E. Dudeney, "Perplexities. With some easy puzzles for beginners. 146.— Water, Gas, and Electricity." *The Strand* **46** (1913), 110, 221.

[5] Ernst Steinitz, *Polyeder und Raumeinteilungen* (Leipzig: Teubner, 1916), 139 pages. [Heft 9 of *Encyclopädie der mathematischen Wissenschaften* **III-1B**.]

[6] Lajos Szilassi, "Regular toroids," *Structural Topology* #13 (1986), 69–80.

[7] `http://cs.stanford.edu/~knuth/SAPsculpture-lores.jpg`, public art at 3408 Hillview Avenue, Palo Alto, California, as photographed by D. E. Knuth on 24 April 2020. [Can anybody help me identify the sculptor?]